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PROGRESS REPORT.

1. *Mr. S. R. Ranganathan, B. A.*—Honours student, Madras Christian College, 22, Kavarai Street, Saidapet, has been elected a member at the concessional rate.

2. Our President Mr. E. W. Middlemast, M.A., Principal and Professor of Mathematics, Presidency College, Madras, being unavoidably absent from India, the Committee have, under Act VIII(*e*) of Constitution, after mature consideration, appointed Dewan Bahadur R. Ramachandra Rao, B.A., Collector, Nellore, to be the President of the Society during the absence of Mr. Middlemast.

3. Our Treasurer Rev. C. Pollard, now in England, having resigned his appointment of the Treasurership, the Committee have confirmed Mr. S. Narayana Aiyar, M.A., F.S.S, as the Treasurer of the Society.

4. The Committee have also thought it desirable to keep the Constitution of the Committee the same as what was notified in the Progress Report for December 1914.

POONA,
31st May 1915.

D. D. KAPADIA,
Hony. Joint Secretary.

A Special Class of Equations.

By M. T. Naraniengar.

Consider the equations

$$\phi(x)-x=0,\ \phi\{\phi(x)\}-x=0,\ \phi[\phi\{\phi(x)\}]-x=0, \text{ etc.}$$

These may conveniently be written

$$\phi(x)-x=0,\ \phi^2(x)-x=0,\ \phi^3(x)-x=0, \text{ etc.}$$

We shall call the *first* equation the *primary* equation of the class, and the rest *secondary* equations.

Theorem I:—Every root of the primary equation is a root of a secondary equation.

For, if α is a root of the primary,

$$\phi(\alpha)=\alpha.$$

$$\therefore \quad \phi^2(\alpha)=\phi[\phi(\alpha)]=\phi(\alpha)=\alpha\ ;$$

or α is also a root of $\phi^2(x)-x=0$; and so on.

Cor. Any root of $\phi^n(x)-x=0$, is also a root of $\phi^{mn}(x)-x=0$.

Theorem II:—If m and n are prime to each other, the equations $\phi^m(x)-x=0$, and $\phi^n(x)-x=0$ have no roots common other than those of the primary.

Find integers a, b such that $am-bn=\pm 1$, which is always possible since m and n are prime to each other.

Now, if α be a common root of the two secondaries,

$$\phi^{am}(\alpha)=\alpha \text{ and } \phi^{bn}(\alpha)=\alpha,$$

supposing $am-bn=1$,

$$\phi^{am}(\alpha)=\phi^{bn+1}(\alpha)=\phi\{\phi^{bn}(\alpha)\}=\phi(\alpha).$$

$$\therefore \quad \phi(\alpha)=\alpha.$$

Hence α is a root of the primary.

Theorem III:—If h is the greatest common factor of m and n, the roots of $\phi^h(x)-x=0$, are all the common roots of $\phi^m(x)-x=0$ and $\phi^n(x)-x=0$.

For, in this case, we can find a, b such that

$$am-bn=\pm h,$$

so that if α is a common root,

$$\phi^{am}(\alpha)=\alpha \text{ and } \phi^{bn}(\alpha)=\alpha.$$

Thus $\phi^{am}(\alpha)=\phi^{bn+h}(\alpha)=\phi^h\{\phi^{bn}(\alpha)\}=\phi^h(\alpha)$.

$$\therefore \quad \phi^h(\alpha)=\alpha.$$

Hence the theorem.

Theorem IV:—If n be a prime number and α be a primitive root of $\phi^n(x)-x=0$, [that is, a root other than the roots of the primary equation], then $\phi(\alpha)$, $\phi^2(\alpha)$,......$\phi^{n-1}(\alpha)$ are also roots and are all different.

Let $\beta_p\equiv\phi^p(\alpha)$, $[p=1, 2, \ldots n-1]$. Then we have

$$\phi^n(\beta_p)=\phi^n\{\phi^p(\alpha)\}=\phi^p\{\phi^n(\alpha)\}=\phi^p(\alpha)=\beta_p.$$

Hence β_p is a root of the equation $\phi^n(x)-x=0$.

Also β_p cannot be equal to β_q, since this would mean

$$\phi^p(\alpha)=\phi^q(\alpha),$$

i.e. $$\phi^{p-q}(\beta_q)=\beta_q, (p>q).$$

In other words, β_q is a common root of $\phi^{p-q}(x)-x=0$ and $\phi^n(x)-x=0$, which is contrary to Theorem II since $(p-q)$ is prime to n.

Theorem V:—If n is a prime number, then

$$[\phi^n(x)-x]\equiv \{\phi(x)-x\} . \prod \{f(x)\}$$

where $f(x)$ is a cyclic expression of the nth degree.

Let $\phi(x)-x=0$ be of degree p. Then the equation $\phi^n(x)-x=0$ is of degree p^n and has (p^n-p) primitive roots, being the roots of

$$\{\phi^n(x)-x\}\div\{\phi(x)-x\}=0. \quad \ldots \quad \ldots \quad (1)$$

But, by Theorem IV, a primitive root α has $(n-1)$ other roots associated with it, *viz.*: $\phi(\alpha)$, $\phi^2(\alpha)$,...$\phi^{n-1}(\alpha)$. Thus the set of roots corresponding to a primitive root (α) belong to a cyclic equation $f(x)=0$ of degree n.

Hence, the left hand side of (1) is made up of factors of the type $f(x)$, which is the result stated in the theorem.

Cor. 1. *Fermats' Theorem follows at once from the above.*

For $$p^n-p\equiv 0, (\text{mod } n).$$

If p and n be prime to each other, therefore,

$$p^{n-1}-1\equiv 0, (\text{mod } n),$$

which is Fermat's Theorem.

Cor. 2. $f\{\phi(x)\}\equiv(-1)^{p-1}f(x_1).f(x_2)......f(x_p)$, where $x_1,x_2,...x_p$ are the p values of X in the equation

$$\phi(X)-\phi(x)=0.$$

For, from the cyclic property of $f(x)$ we readily see that $f\{\phi(x)\}$ is divisible by $f(x)$; so that, the left hand member is divisible by the product $f(x_1)$, $f(x_2)......f(x_p)$. Also, since

$$x_1\,x_2...x_p=(-1)^{p-1}\phi(x),$$

and $f(x)$ is of the nth degree, the coefficient must be $(-1)^{n(p-1)}$ which is the same as $(-1)^{p-1}$, n being a prime number.

An Example.

Suppose $\phi(x)\equiv x^3$, and $n=5$. From Theorem V we deduce that

$$(x^{81}-1)\equiv(x-1)\prod \{f(x)\}$$

where $f(x)$ is a cyclic expression of the fifth degree. We shall now proceed to examine the form of $f(x)$ in detail.

1. Consider the equation

$$x^5-p_1x^4+p_2x^3-p_3x^2+p_4x-p_5=0 \quad \ldots \quad \ldots \quad (I)$$

whose roots are $\alpha, \alpha^2, \alpha^4, \alpha^8, \alpha^{16}$. $(\alpha^{31}=1)$.

Put

$$y_1=\alpha+\alpha^2+\ldots\alpha^{16}$$
$$y_2=\alpha^3+\alpha^6+\ldots\alpha^{48}(=\alpha^{17})$$
$$y_3=\alpha^9+\alpha^{18}+\ldots\alpha^{20}$$
$$y_4=\alpha^{27}+\alpha^{23}+\ldots\alpha^{29}$$
$$y_5=\alpha^{19}+\alpha^7+\ldots\alpha^{25}$$
$$y_6=\alpha^{26}+\alpha^{21}+\ldots\alpha^{13},$$

and $z_1=y_1+y_4$; $z_2=y_2+y$; $z_3=y_3+y_6$.

Then we easily find
$$\left.\begin{aligned}y_1y_2&=y_1+y_3+2y_5+y_6\\ y_1y_3&=y_2+y_3+y_5+2y_6\end{aligned}\right\}$$

$$y_1y_4=z_1+z_2+5\text{ ; } y_2y_5=z_2+z_3+5\text{ ; } y_3y_6=z_3+z_1+5$$
$$z_1z_2=4z_1+3z_2+3z_3\text{ ; \&c. ; } z_1^2=3z_1+4z_2+2z_3+10.$$

Thus
$$\left.\begin{aligned}\Sigma(z_1)&=\Sigma(y_1)=-1\\ \Sigma(z_1z_2)&=10\Sigma(z_1)=-10\\ z_1z_2z_3&=32\Sigma(z_1)+40=8\end{aligned}\right\},$$

so that z satisfies the equation

$$z^3+z^2-10z-8=0. \quad \ldots \quad \ldots \quad \text{(II)}$$

Hence y_1, y_4 are the roots of

$$y^2-z_1y+z_1+z_2+5=0\text{ ; \&c.}$$

Hence the equation whose roots are y_1, y_2, y_3, y_4, y_5, y_6 is

$$\Pi\ (y^2-z_1y+z_1+z_2+5)=0,$$

or
$$y^6+y^5+3y^6+11y^3+44y^2+36y+32=0, \quad \ldots \quad \ldots \quad \text{(III)}$$

by virtue of (II)

Also, from (I),

$$p_1=y_1\text{ ; } p_2=y_2+y_3\text{ ; } p_3=y_5+y_6\text{ ; } p_4=y_4\text{ ; } p_5=1.$$

Thus, finally, we have the following:

The cyclic equation

$$x^5-y_1x^4+(y_2+y_3)x^3-(y_5+y_6)x^2+y_4x-1=0$$

where y_1, $y_2 \ldots y_6$ are the roots of (III), has for roots $\alpha, \alpha^2, \alpha^4, \alpha^8, \alpha^{16}$; and

$$\frac{x^{31}-1}{x-1}\equiv\Pi\ \{\ x^5-y_1x^4+\ldots-1\ \}.$$

2. Equation (III) may be directly obtained as follows :—

Write down the values of y_1^2, y_1y_2, ...as linear expressions in terms of y_1, y_2, ...in a tabular form.

Thus we have

	y_1	y_2	y_3	y_4	y_5	y_6	
y_1^2	1	2	2	...	...	...	
y_1y_2	1	...	1	...	2	1	
y_1y_3	...	1	1	...	1	2	
y_1y_4	1	1	...	1	1	...	5
y_1y_5	1	1	...	2	1	...	
y_1y_6	...	1	...	2	1	1	

From the above table we get the values of y_1^3, y_1^4, y_1^5, y_1^6 at once as given below.

	y_1	y_2	y_3	y_4	y_5	y_6	
y_1^3	3	4	6	...	6	6	
y_1^4	13	24	16	24	26	22	
y_1^5	87	114	66	120	136	78	120
y_1^6	577	574	354	548	628	324	600

[Multiply the first three rows of the first table by 1, 2, 2 respectively and add; the result is y_1^3. Multiply the rows by 3, 4, 6, 0, 6, 6 in order and add to get y_1^4; and so on.]

From these we deduce

$s_1=\Sigma(y_1)=-1$; $s_2=\Sigma(y_1^2)=+5\ \Sigma(y_1)=-5$;

$s_3=\Sigma(y_1^3)=25\Sigma(y_1)=-25.$

$s_4=-125$; $s_5=-601+720=119$; $s_6=-3005+3600=595.$

Hence by Newton's formulae

$p_1=-1.$

$-2p_2=-5-1=-6. \quad \therefore p_2=3.$

$3p_3=-25-5-3=-33. \quad \therefore p_3=11.$

$-4p_4=-125-25-15-11=-176. \quad \therefore p_4=44.$

$5p_5=119-125-75-55-44=-180. \quad \therefore p_5=36.$

$-6p_6=595+119-375-275-220-36=-192. \quad \therefore p_6=32.$

The equation in y is thus established, and is identical with (III).

3. The co-efficients in equation (III) may also be calculated by arranging the symmetric functions of the y's into groups.

Thus

$$\Sigma(y_1y_2) = (12) + (13) + \tfrac{1}{2}(14),$$

where (12) denotes $(y_1y_2 + y_2y_3 + y_3y_4 + y_4y_5 + y_5y_6 + y_6y_1)$

(13) " $(y_1y_3 + y_2y_4 + y_3y_5 + y_4y_6 + y_5y_1 + y_6y_2)$

(14) " $(y_1y_4 + y_2y_5 + y_3y_6 + y_4y_1 + y_5y_2 + y_6y_1)$

Similarly

$$\Sigma(y_1y_2y_3) = (123) + (124) + (134) + \tfrac{1}{3}(135)$$
$$\Sigma(y_1y_2y_3y_4) = (1234) + (1235) + \tfrac{1}{2}(1245)$$
$$\Sigma(y_1y_2y_3y_4y_5) = (12345)$$

By means of the first table the several cyclic groups can be linearly expressed in terms of the y's, and the values of the symmetric functions found by substitution. The process, however, is tedious.

4. Returning to equation (I), write

$$y_1 = \alpha + \alpha^{26} + \alpha^{25} + \alpha^{30} + \alpha^{5} + \alpha^{6}$$
$$y_2 = \alpha^{3} + \alpha^{16} + \alpha^{13} + \alpha^{28} + \alpha^{15} + \alpha^{18}$$
$$y_3 = \alpha^{9} + \alpha^{17} + \alpha^{8} + \alpha^{22} + \alpha^{14} + \alpha^{23}$$
$$y_4 = \alpha^{27} + \alpha^{20} + \alpha^{24} + \alpha^{4} + \alpha^{11} + \alpha^{7}$$
$$y_5 = \alpha^{19} + \alpha^{29} + \alpha^{20} + \alpha^{12} + \alpha^{2} + \alpha^{24}$$

following Gauss's method; we get

	y_1	y_2	y_3	y_4	y_5	
y^2	2	0	0	2	1	6
y_1y_2	0	1	2	1	2	
y_1y_3	0	2	2	1	1	
y_1y_4	2	1	1	0	2	
y_1y_5	1	2	1	2	0	

Proceeding as before, we find

y_1^3	15	4	3	6	6	12
y_1^4	60	28	26	49	38	90
y_1^5	346	205	195	250	240	360

Hence, we can at once write

$$s_1=p_1=-1.$$
$$s_2=\Sigma(y_1^2)=30-5=25.$$
$$s_3=\Sigma(y_1^3)=60-34=26.$$
$$s_4=\Sigma(y_1^4)=450-201=249.$$
$$s_5=\Sigma(y_1^5)=1800-1236=564.$$

And, from Newton's formulae, we deduce

$$p_2=-12,$$

$$3p_3=26+25+12=63. \quad \therefore\ p_3=21.$$

$$-4p_4=249+26-300+21=-4. \quad \therefore\ p_4=1.$$
$$5p_5=564+249-312-525-1=-125. \quad \therefore\ p_5=-25.$$

Hence the equation, whose roots are the y's, is

$$y^5+y^4-12y^3-21y^2+y+25=0. \quad \ldots \quad \ldots \quad \text{(i)}$$

In other words,

$$(x^{31}-1/(x-1)\equiv\Pi\{x^6-y_1x^5+(y_1+y_4+3)x^4-(2y_1+y_5+2)x^3 + (y_4+y_1+3)x^2-y_1x+1\}.$$

where the y's are the roots of (i).

5. Lastly, the method of transformation may be employed to reach the same result. Thus, putting

$$f(x)\equiv x^5-p_1x^4+p_2x^3-p_3x^2+p_4x-p_5=0,$$

in § **1** above, we obtain the identity

$$f(x^2)\equiv -f(x).\ f(-x);$$

whence, by equating co-efficients,

$$\left.\begin{array}{l} 2p_2=p_1^2-p_1;\ p_2^2-p_2=2(p_1p_3-p_4); \\ 2p_3=p_4^2-p_4;\ p_3^2-p_3=2(p_4p_2-p_1); \end{array}\right\}$$
$$p_5=1.$$

Also, giving x the value **1**, we have

$$p_1+p_2+p_3+p_4+1=0.$$

Write $p_1+p_4=z$, so that $p_2+p_3+z+1=0$; then eliminating p_2, p_3,

$$p_1^2+p_4^2+p_1+p_4+2=0,$$
$$z^2+z+2=2p_1p_4.$$

Again $p_4(p_2^2-p_2)-p_1(p_3^2-p_3)=2p_1p_4(p_3-p_2)-2(p_4^2-p_1^2)$, whence substituting for p_2, p_3 and dividing out by (p_1-p_4), we get

$$p_1p_4(p_1^2+p_1p_4+p_4^2)+2p_1p_4(p_1+p_4)-5p_1p_4-8(p_1+p_4)=0.$$

i.e.,
$$z^4+4z^3-7z^2-38z-24=0.$$

Rejecting the factor $(z+3)$* we are thus led to the cubic

$$z^3+z^2-10z-8=0,$$

which is the same a sequation (II).

The values of p_1, p_4 are readily found as the roots of the quadratic

$$y^2-zy+\tfrac{1}{2}(z^2+z+2)=0,$$

since $p_1+p_4=z$, and $p_1p_4=\tfrac{1}{2}(z^2+z+2)$.

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* The factor $(z+3)$ is rejected, because if $z=-3$, p_1 and p_4 are the roots of

$$p^2+3p+4=0;$$

and the quintic in this case becomes

$$x^5-px^4-2(p+1)x^3-2(p-2)x^2-(p+3)x-1=0,$$

which is the same as

$$(x^2+x+1)\{x^3-(p+1)x^2-(p+2)x-1\}=0.$$

The cubic factor here corresponds to the roots α, α^2, α^4 of the binomial equation $x^7-1=0$.

Mathematical Training from an Economic Point of View * (An Extract.)

Some one has said "a man's capacity for mathematics is paralleled by the development of the mathematical activity present in the great departments of investigation." Formerly the subject of biology was considered remotely distinct and separated from mathematical analysis. Then those students, who were not inclined to the mathematical discipline of their courses flocked to biologic studies only to learn that they had chosen intensely mathematical subjects.

It seems to me that our great work as mathematics teachers will be to lend our influence to the discovery of the relationships which underlie other fields of research. If these relationships belong to any mathematical category, as I believe they do, then we have a large problem whose solution shall not only conserve the mathematical field but shall add the greatest stimulus to mathematical training.

Let us determine in how far each of the arbitrary divisions of the subject matter of mathematics contributes to desirable human achievements. Let us give proportionate weights to these divisions which we designate as *algebra*, *geometry*, *analysis* and *mechanics*, and determine whether the time spent upon them is justified, or whether the raw product, the material of our text-books, is poor, or whether we are extravagant or indulgent in the luxuries of our subject. It is generally conceded that mathematics knows no poverty. If therefore we should discover that our arbitrary division of mathematics is wrong, let us have the courage to throw it out. We must recognize the cold hard fact that we are dealing with a commercial problem in which we shall be harder pressed. If we can give no adequate answer, then we are no better than the unskilled labourer in the street. The modern tendency in education is democratic, not aristocratic as men have vainly attempted to make it in the past. Our ranks are swollen with men and women whose ideals and culture are no higher than those of despised workers in other vocations. Professional life is no longer a guarantee of the homage of exaltation. Why is it so? Because we have introduced so much in our studies that is unreal. We have deceived ourselves into believing that our efficiency was measured by the number of students graduated or passed to a higher grade. What about all those young men and women who have been neglected because we thought their withdrawal from school or college was an expression of unfitness? Yes, an unfitness in most cases for an insufficient school system. Now

* From a paper by Mr. E. H. Koch Jr., of New York.

as mathematics teachers how heavily have we played in the *role.* We must not be amazed then if we hear a challenge put on our cherished subjects. Will algebra, geometry, trigonometry, analytics, calculus, even arithmetic, stand the test when we ask without bias? Are they taught efficiently in their present text-book form?" They have served us well in the past, but had we not better tear them down and build new structures in their places? If we are content to let them remain for a longer time, we must repair and replenish them. Most of our teachers look upon algebra, geometry and trigonometry as if they were the body of materials embodied in a standard text-book on that subject. We have just as many blind teachers who regard the formula as wooden as if it were intended merely for substitution from data instead of for its larger purpose of interpretation. These impediments to efficiency in mathematical teaching are the result of our failure to recognize the world as a domain of relationships. It does seem timely that we should incorporate all these mathematical subjects and teach them more efficiently as an organized whole. We can teach mathematics more efficiently by presenting 'theory' through the problem and postponing the former until we need it and can make immediate use of it. In this way we can avoid a great amount of the drudgery of algebra.

"As each subject grows it fringes" and in the garden of mathematics there are weeds to pull out. One of the most persistent weeds that grow among the flowers in our estate is the so-called "problem." Why do we permit these monstrosities, these unreal problems to fill our text-books? They represent a type of fiction which is revolting to the pupil. They are vicious in principle because they contradict the realities of life. They negate that highest virtue of mathematics, the search for truth. Every puzzle, or "faked" or "unreal" problem inserted in our text-books increases the denominator of the fraction or ratio of 'efficiency' and therefore decreases its value. We should establish the principle never to insert a problem which has to be worked backwards in contradistinction to the realities of life. There is plenty of good material for real live problems, but if our knowledge of real things is too limited we had better omit the fictitious problems entirely.

It is a disgrace in this time and age to continue to suffer under the yoke of the authority of such text-books, and in the name of the thousands of students who are preparing for college and especially the technical school we protest against this infliction.

There has been reason for honest variance of opinion regarding the cultural and utilitarian value of mathematics as long as these claims, remained entrenched in the old undisputed doctrines: "There is no

subject labelled cultural which is not at the same time practical and there is no subject labelled practical which is not at the same time cultural." Through mutual understanding we realize that education is beginning to teach subject-matter in terms of daily life and is recognizing the factors which are of value to our people. We have made the first serious attempt in the history of our education to meet the demand of our economic, industrial and social problems. We have learned that tremendous industrial forces have been developed without any commensurate adjustment of our school system. We may no longer disregard the significance of industry in its influence upon our life. Our systems and subject matter must be rehabilitated. Out of the prevailing systems there have emerged forms of practical mathematics which connect the school with the industries and vocations of our people. They involve a more or less special training which is best adapted to make the student discover for himself and to promote the qualities essential to his success in a specific vocation. We are getting a little closer to the various types of mind and the precise factors of intelligence.

Two years ago the speaker undertook the unification of the essentials of elementary mathematics for electrical students. The subject matter of this practical mathematics extends over the usual range of elementary mathematics including the notions and symbolism of the elements of calculus and vector analysis and vector applications. This was necessary in order to meet the students' requirements in both direct and alternating current work. All non-essentials are omitted and the teaching of theory is postponed until the student requires it in his work. We count this a great gain in economy. The student has a clear understanding of his work. He can express ideas intelligently. He has the power of applying knowledge practically. He realizes the right thing to do in an emergency. He accomplishes a reasonable amount of work in a definite time. His class room is always a work room. His work is interesting. It gives him pleasure. He has greater physical endurance, poise, displays greater fidelity, assurance, neatness, accuracy and the greatest of all blessings 'common-sense.' He is also trained for leadership and executive positions by imposing upon him responsibility of foremanship work in the class room, laboratory and shop. Upon entering the class room he immediately reports his outsigned work on a time sheet on the front cover of his work-book and makes a similar entry at the close of the period. The work-book contains every scrap of work which the student has completed and this book is retained by the instructor who has the student's entire work under constant supervision and inspection. The work of the instructor is

made more efficient by the co-operation of the students through their foreman. We are simply recognizing a well established principle that a man can learn to direct others by first learning to be guided by others. Secondly, he can learn to guide others best by working with them, sharing their trials and disappointments, and gaining an intimate appreciation of and sympathy with their modes of thought and action. "Of the many evils which our examination system has inflicted upon us the central one has consisted in forcing our school and University teaching into moulds determined not by the true interests of education but the mechanical exigencies of the examination syllabus." †

The time is not far distant when our college and State examination Boards will recognize these facts and they will be compelled to offer examinations in a unified course such as practical mathematics. These demands will come first from the technical schools.

We cannot all be great mathematicians, and so our universities would become more efficient if they did not attempt to duplicate work which could be undertaken better by a few institutions. I wish we would hear the phrase 'institutions for teaching' very more often than we hear about 'institutions for learning.' Would it not be the part of wisdom if many of our institutions devoted their time in discovering how mathematics is contributing to other fields of intellectual activities and thereby promote such projects? We have seen an increased efficiency in mathematics teaching since history stepped in and opened a new pathway. We should establish experiment stations not only in agriculture and engineering, but also in mathematics and other subjects as well. Then the study of higher mathematics will have an immediate justification as well as blazing the future pathway of knowledge.

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† Science, September 23, 1910.

SHORT NOTES.

On the Integral $\int_0^x \frac{\tan^{-1}t}{t}dt$.

1. Let

$$\phi(x)=\int_0^x \frac{\tan^{-1}t}{t}dt. \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

Then it is easy to see that

$$\phi(x)+\phi(-x)=0; \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

and that

$$\phi(x)=\frac{x}{1^2}-\frac{x^3}{3^2}+\frac{x^5}{5^2}-\frac{x^7}{7^2}+\ldots \quad \ldots \quad \ldots \quad (3)$$

provided that $|x|\leq 1$.

Changing t into $1/t$ in (1), we obtain

$$\phi(x)-\phi\left(\frac{1}{x}\right)=\frac{1}{2}\pi\log x, \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

provided that the real part of x is positive.

The results in the following two sections can be very easily proved by differentiating both sides with respect to x.

2. If $0<x>\frac{1}{2}\pi$, then

$$\frac{\sin 2x}{1^2}+\frac{\sin 6x}{3^2}+\frac{\sin 10x}{5^2}+ \quad \ldots \quad \ldots \quad (5)$$

$$=\phi(\tan x)-x\log(\tan x).$$

If, in particular, we put $x=\frac{1}{8}\pi$ and $\frac{1}{12}\pi$ in (5), we obtain

$$\frac{1}{1^2}-\frac{1}{5^2}+\frac{1}{9^2}-\frac{1}{13^2}+\ldots=\frac{1}{\sqrt{2}}\phi(\sqrt{2}-1)+\frac{\pi}{8\sqrt{2}}\log(1+\sqrt{2})+\frac{\pi^2}{16}; \quad \ldots \quad (6)$$

and

$$2\phi(1)=3\phi(2-\sqrt{3})+\frac{1}{4}\pi\log(2+\sqrt{3}). \quad \ldots \quad \ldots \quad (7)$$

If $-\frac{\pi}{2}<x<\pi$, then

$$2\phi\left(\tan\frac{x}{2}\right)=\sin x+\frac{2}{3}\cdot\frac{\sin^3 x}{3}+\frac{2\cdot 4}{3\cdot 5}\cdot\frac{\sin^5 x}{5}+ \quad \ldots \quad (8)$$

If $0<x<\frac{1}{2}\pi$, then

$$\frac{\sin x}{1^2}\cos x+\frac{\sin 2x}{2^2}\cos^2 x+\frac{\sin 3x}{3^2}\cos^3 x+\ldots$$

$$=\phi(\tan x)+\tfrac{1}{2}\pi\log\cos x-x\log\sin x; \ldots \quad (9)$$

and

$$\frac{\cos x+\sin x}{1^2}+\frac{1}{2}\cdot\frac{\cos^3 x+\sin^3 x}{3^2}+\frac{1\cdot 3}{2\cdot 4}\cdot\frac{\cos^5 x+\sin^5 x}{5^2}+\ldots$$

$$=\phi(\tan x)+\tfrac{1}{2}\pi\log(2\cos x). \quad (10)$$

If $-\frac{1}{2}\pi<x<\frac{1}{2}\pi$ and a be any number such that

$$\left|(1-a)\sin x\right|\leq 1, \text{ and } \left|\left(1-\frac{1}{a}\right)\cos x\right|\leq 1, \text{ then}$$

$$\frac{\sin x}{1^2}\left(1-\frac{1}{a}\right)\cos x+\frac{\sin 2x}{2^2}\left(1-\frac{1}{a}\right)^2\cos^2 x+\frac{\sin 3x}{3^2}\left(1-\frac{1}{a}\right)^3\cos^3 x+\ldots$$

$$+\frac{\sin(x+\frac{1}{2}\pi)}{1^2}(1-a)\sin x+\frac{\sin 2(x+\frac{1}{2}\pi)}{2^2}(1-a)^2\sin^2 x+\ldots$$

$$=\phi(\tan x)-\phi(a\tan x)+x\log a. \quad \ldots \quad (11)$$

3. Let R(x) and I(x) denote the real and the imaginary parts of x respectively. Then, if $-1<\text{R}(x)<1$,

$$\log\left(1-\frac{x^2}{1^2}\right)-3\log\left(1-\frac{x^2}{3^2}\right)+5\log\left(1-\frac{x^2}{5^2}\right)-\ldots\ldots$$

$$=\frac{4}{\pi}\left[\phi(1)-\phi\left\{\tan\tfrac{1}{4}\pi(1-x)\right\}\right]+\log\tan\tfrac{1}{4}\pi(1-x). \quad (12)$$

Putting $x=\frac{2}{3}$ in (12) and using (7), we obtain

$$\left(1-\frac{4}{3^2}\right)\left(1-\frac{4}{9^2}\right)^{-3}\left(1-\frac{4}{15^2}\right)^5\left(1-\frac{4}{21^2}\right)^{-7}\left(1-\frac{4}{27^2}\right)^9\ldots$$

$$=(2-\sqrt{3})^{\frac{2}{3}}e^n, \text{ where } n=\frac{4}{3\pi}\phi(1). \quad (13)$$

Again subtracting $\log(1-x)$ from both sides in (12) and making $x\to 1$, we obtain

$$\left(1-\frac{1}{3^2}\right)^{-3}\left(1-\frac{1}{5^2}\right)^5\left(1-\frac{1}{7^2}\right)^{-7}\left(1-\frac{1}{9^2}\right)^9\ldots\ldots$$

$$=\frac{\pi}{8}e^{\frac{4}{\pi}\phi(1)}. \quad (14)$$

If $-1<\text{I}(x)<1$, then

$$\log\left(1+\frac{x^2}{1^2}\right)-3\log\left(1+\frac{x^2}{3^2}\right)+5\log\left(1+\frac{x^2}{5^2}\right)-\ldots$$

$$=\frac{4}{\pi}\left\{\phi(1)-\phi\left(e^{-\frac{1}{2}\pi x}\right)\right\}2x\tan^{-1}e^{-\frac{1}{2}\pi x}. \quad (15)$$

From this and (7), we see that if $\frac{1}{2}\pi x=\log(2+\sqrt{3})$, then

$$\left(1+\frac{x^2}{1^2}\right)^1\left(1+\frac{x^2}{3^2}\right)^{-3}\left(1+\frac{x^2}{5^2}\right)^5\left(1+\frac{x^2}{7^2}\right)^{-7}\ldots\ldots=e,\frac{4}{3\pi}\phi(1), \quad (16)$$

where n is the same as in (13).

It follows at once from (12) and 15 that, if

$$-1<\text{R}(\beta)<1, -1<\text{I}(\alpha)<1,$$

then

$$e^{\frac{1}{2}\pi\alpha\beta}=\left(\frac{1^2+\alpha^2}{1^2-\beta^2}\right)^1\left(\frac{3^2-\beta^2}{+\alpha^2}\right)^3\left(\frac{5^2+\alpha^2}{5^2-\beta^2}\right)^5\left(\frac{7^2-\beta^2}{7^2+\alpha^2}\right)^7\ldots \quad (17)$$

provided that $\cosh\frac{1}{2}\pi\alpha=\sec\frac{1}{2}\pi\beta$.

4. Now changing x into $2x(1+i)$ in (15), we have

$$\log\left(1+\frac{8ix^2}{1^2}\right)-3\log\left(1+\frac{8ix^2}{3^2}\right)+5\log\left(1+\frac{8ix^2}{5^2}\right)-\ldots\ldots$$
$$=\frac{4}{\pi}\phi(1)-4x(1+i)\ \tan^{-1}e^{-\pi x(1+i)}$$
$$-\frac{4}{\pi}\left\{\frac{1}{1^2}e^{-\pi x(1+i)}-\frac{1}{3^2}e^{-\pi x(1+i)}+\ldots\right\}$$

Equating real and imaginary parts, we see that if x is positive, then

$$\log\left(1+\frac{64x^4}{1^4}\right)-3\log\left(1+\frac{64x^4}{3^4}\right)+5\log\left(\frac{64x^4}{1+5^4}\right)-\ldots$$
$$=\frac{8}{\pi}\phi(1)-2x\ \log\left(\frac{\cosh\ \pi x+\sin\ \pi x}{\cosh\ \pi x-\sin\ \pi x}\right)-4x\ \tan^{-1}\left(\frac{\cos\ \pi x}{\sinh\ \pi x}\right)$$
$$-\frac{8}{\pi}\left\{\frac{\cos\ \pi x}{1^2}e^{-\pi x}-\frac{\cos 3\pi x}{3^2}e^{-3\pi x}+\frac{\cos 5\pi x}{5^2}e^{-5\pi x}-\ldots\right\};\quad (18)$$

and

$$\tan^{-1}\frac{8x^2}{1^2}-3\ \tan^{-1}\frac{8x^2}{3^2}+5\ \tan^{-1}\frac{8x^2}{5^2}-$$
$$=x\log\left(\frac{\cosh\ \pi x+\sin\ \pi x}{\cosh\ \pi x-\sin\ \pi x}\right)-2x\ \tan^{-1}\left(\frac{\cos\ \pi x}{\sinh\ \pi x}\right)$$
$$+\frac{4}{\pi}\left\{\frac{\sin\ \pi x}{1^2}e^{-\pi x}-\frac{\sin 3\pi x}{3^2}e^{-3\pi x}+\frac{\sin 5\pi x}{5^2}e^{-5\pi x}-\ldots\right\}\quad (19)$$

It follows from (18) that, if n is a positive odd integer, then

$$\left(1+\frac{4n^4}{1^4}\right)\left(1+\frac{4n^4}{3^4}\right)^{-3}\left(1+\frac{4n^4}{5^4}\right)^{5}\left(1+\frac{4n^4}{7^4}\right)^{-7}\ldots\ldots$$
$$=e^{\frac{8}{\pi}\phi(1)}\left(\frac{1-e^{-\frac{1}{2}\pi n}}{1+e^{-\frac{1}{2}\pi n}}\right)^{2n\cos\frac{1}{2}(n-1)\pi}\quad (20)$$

and if n is any even integer then

$$\left(1+\frac{4n^4}{1^4}\right)\left(1+\frac{4n^4}{3^4}\right)^{-3}\left(1+\frac{4n^4}{5^4}\right)^{5}\left(1+\frac{4n^4}{7^4}\right)^{-7}\ldots\ldots\quad\ldots\quad (21)$$
$$=\exp\left\{\frac{8}{\pi}\phi(1)-\frac{8}{\pi}(-1)^{\frac{1}{2}n}\left(\phi\left(e^{-\frac{1}{2}\pi a}\right)+\tfrac{1}{2}\pi n\ \tan^{-1}e^{-\frac{1}{2}\pi n}\right)\right\}$$

Similarly from (19) we see that, if n is any positive odd integer then

$$\tan^{-1}\frac{2n^2}{1^2}-3\ \tan^{-1}\frac{2n^2}{3^2}+5\ \tan^{-1}\frac{2n^2}{5^2}-\ldots\ldots$$
$$=\frac{4}{\pi}(-1)^{\frac{1}{2}(n-1)}\left\{\frac{\pi n}{4}\log\left(\frac{1+e^{-\frac{1}{2}\pi n}}{1-e^{-\frac{1}{2}\pi n}}\right)+\frac{1}{1^2}e^{-\frac{1}{2}\pi n}\right.$$
$$\left.+\frac{1}{3^2}e^{-\frac{3}{2}\pi n}+\frac{1}{5^2}e^{-\frac{5}{2}\pi n}+\ldots\ldots\right\};\quad (22)$$

and, if n is a positive even integer, then

$$\tan^{-1}\frac{2n^3}{1^3}-3\tan^{-1}\frac{2n^3}{3^3}+5\tan^{-1}\frac{2n^3}{5^3}-\ldots\ldots$$

$$=2n(-1)^{\frac{1}{2}n-1}\tan^{-1}e^{-\frac{1}{2}\pi n}. \qquad (23)$$

In this connection it may be interesting to note that

$$\tan^{-1}e^{-\frac{1}{2}\pi n}=\frac{\pi}{4}-\left(\tan^{-1}\frac{n}{1}-\tan^{-1}\frac{n}{3}+\tan^{-1}\frac{n}{5}-\ldots\ldots\right\} \quad \ldots \quad (24)$$

for all real values of n.

5. Remembering that

$$\frac{\pi}{4\cosh\pi x}=\frac{1}{1^2+4x^2}-\frac{3}{3^2+4x^2}+\frac{5}{5^2+4x^2}-\ldots\ldots$$

we have

$$\frac{\pi}{4}\sum_{1}^{\infty}\frac{1}{n^2\cosh\pi nx}=\sum_{n=1}^{n=\infty}\left\{\frac{1}{n^2(1^2+4n^2x^2)}-\frac{3}{n^2(3^2+4n^2x^2)}+\ldots\ldots\right\}$$

$$=\frac{\pi^3}{16}\left(\frac{1}{3}+\frac{x^2}{2}\right)-\frac{\pi x}{2}\left(\frac{\coth\frac{\pi}{2x}}{1^2}-\frac{\coth\frac{3\pi}{2x}}{3^2}+\frac{\coth\frac{5\pi}{2x}}{5^2}-\ldots\ldots\right) \qquad (25)$$

That is to say, if α and β are real and $\alpha\beta=\pi^2$, then

$$\phi(1)+2\phi(e^{-\alpha})+2\phi(e^{-2\alpha})+2\phi(e^{-3\alpha})+\ldots \quad \ldots$$

$$=\frac{\pi}{8}\left(\frac{\alpha}{3}+\frac{\beta}{2}\right)-\frac{\pi}{4\beta}\left\{\frac{1}{1^3\cosh\beta}+\frac{1}{2^3\cosh 2\beta}+\ldots\right\} \qquad (26)$$

If, in particular, we put $\alpha=\beta=\pi$ in (26), we obtain

$$\phi(1)=\frac{5\pi^2}{48}-2\left\{\frac{1}{1^2(e^{\pi}-1)}-\frac{1}{3^2(e^{3\pi}-1)}+\frac{1}{5^2(e^{5\pi}-1)}\ldots\right\}$$

$$-\frac{1}{2}\left\{\frac{1}{1^3(e^{\pi}+e^{-\pi})}+\frac{1}{2^3(e^{2\pi}+e^{-2\pi})}+\frac{1}{3^3(e^{3\pi}+e^{-3\pi})}+\ldots\right\}$$

$$=\cdot 9\,1\,5\,9\,6\,5\,5\,9\,4\,2, \qquad (27)$$

approximately.

S. RAMANUJAN.

A Theorem on Areas.

1. Let a number of particles $a_1, a_2, \ldots a_n$ of masses $m_1, m_2, \ldots m_n$ move in any manner in a plane and return to their initial positions *simultaneously*, after describing curves of areas $A_1, A_2, \ldots A_n$. Then the area G described by their centroid g is given by

$$\Sigma(m_1).G=\Sigma(m_1A_1)-\Sigma(m_1R_1),$$

where $R_1, R_2, \ldots$ are the areas enclosed by the paths of $a_1, a_2, \ldots$ relative to g.

For, we have

$$A=\int(y\,dx),\ G=\int(\bar{y}\,d\bar{x}),\ R=\int\{(y-\bar{y})\,d(x-\bar{x})\};$$
$$\bar{x}=\Sigma(mx)/\Sigma(m),\ \bar{y}=\Sigma(my)/\Sigma(m),\ d\bar{x}=\Sigma(m\,dx)/\Sigma(m),\ \&c.$$

$$\therefore\ \Sigma(mR)=\Sigma\int\{m(y-\bar{y})\,d(x-\bar{x})\}$$
$$=\Sigma(m\int y\,dx)+\Sigma(m\int\bar{y}\,d\bar{x})-\Sigma(m\int y\,d\bar{x})-\Sigma(m\int\bar{y}\,dx)$$
$$=\Sigma(mA)+\Sigma(mG)-\Sigma(\int my\,d\bar{x})-\int\bar{y}\Sigma(m\,dx)$$
$$=\Sigma(mA)+\Sigma(m).G-\Sigma(m).G-\Sigma(m).G$$
$$=\Sigma(mA)-\Sigma(m).G.$$

Hence $\qquad \Sigma(m).\,G=\Sigma(mA)-\Sigma(mR).$

Cor. 1. In particular, if the particles move as a rigid system, the relative paths are all circles; we therefore have

$$\Sigma(m).G=\Sigma(mA)-\pi.I$$

where I is the moment of inertia of the system about g.

Cor. 2. If the particles are free to move in a line, while the line itself moves in any manner parallel to itself, the relative paths are segments of straight lines and $R=0$. The formula becomes

$$\Sigma(m).G=\Sigma(m.A).$$

2. Consider three particles a_1, a_2, a_3, moving as a rigid system and describing equal areas A. Here

$$G=A-\pi.k^2$$

where k = radius of gyration of the particles round g.

Also, if r be the circumradius of the invariable triangle $a_1\,a_2\,a_3$, and o the circumcentre

$$k^2=r^2-og^2,$$

whence

$$G=A-\pi r^2+\pi og^2.$$

Thus, all points g in the plane of $a_1\,a_2\,a_3$ which describe equal areas (as a_1, a_2, a_3 describe equal areas A) lie on a circle with centre o; the area described by o is $(A-\pi r^2)$ and is the minimum area for points in the plane.

Cor. 1. If x, y, z, be the areal co-ordinates of g with respect to $a_1\,a_2\,a_3$, the above result can be written

$$G=xA_1+yA_2+zA_3+\pi t^2,$$

where t is the tangent from g to the circumcircle.

Cor. 2. On any straight line there is one point for which the area described is less than for any other point: this point being the foot of the perpendicular from o on the line.

If p, q, r be three collinear points, then

$$qr.P+rp.Q+pq.R+\pi.pq.qr.rp=0,$$

since $P=A-\pi r^2+\pi.op^2$, &c.

3. In the previous article, let circles be described round each point of the plane equal in area to the area described by the point. Then, *the radical axis of any two circles of the system passes through the circumcentre* o.

For, if p_1, p_2 be two points and ρ_1, ρ_2 the radii of the corresponding circles, we have

$$P_1=\pi\ \rho_1^2=A-\pi\ r^2+\pi op_1^2,$$
$$P_2=\pi\ \rho_2^2=A-\pi\ r^2+\pi.op_2^2$$
$$\therefore\quad op_1^2-op_2^2=\rho_1^2-\rho_2^2.$$

Cor. Any three circles drawn as above have the minimum point o for their radical centre; and all the circles have a common orthogonal circle.

R. Vythynathaswamy.

SOLUTIONS.

Question 358.

(S. RAMANUJAN) :—If n is a multiple of 4 excluding zero, shew that

$$1^{n-1}\operatorname{sech}\frac{\pi}{2}-3^{n-1}\operatorname{sech}\frac{3\pi}{2}+5^{n-1}\operatorname{sech}-+\ldots=0.$$

Solution by M. Bhimasena Rao.

We know

$$\int_0^\infty \frac{2\cos 2xt}{\cosh \pi t}\,dt=\operatorname{sech} x.$$

$$\therefore \int_0^\infty \frac{2\sin xt.\,\sin at}{\cosh \pi t}\,dt=\int_0^\infty \frac{\cos(x-a)t-\cos(x+a)t}{\cosh \pi t}\,dt$$

$$=\tfrac{1}{2}\left(\operatorname{sech}\frac{x-a}{2}-\operatorname{sech}\frac{x+a}{2}\right).$$

Also,

$$\int_0^\infty \frac{2\sin at}{\cosh \pi t}.\frac{\sin(4n+2)xt}{2\cos xt}\,dt$$

$$=\int_0^\infty \frac{2\sin at.dt}{\cosh \pi t}(\sin xt-\sin 3xt+\sin 5xt\ldots\ldots\ldots+\sin(4n+1)xt).$$

$$=\tfrac{1}{2}\left\{\operatorname{sech}\frac{x-a}{2}-\operatorname{sech}\frac{3x-a}{2}+\operatorname{sech}\frac{5x-a}{2}\ldots\ldots\ldots\ldots\right\}$$

$$-\tfrac{1}{2}\left\{\operatorname{sech}\frac{x+a}{2}-\operatorname{sech}\frac{3x+a}{2}+\operatorname{sech}\frac{5x+a}{2}\ldots\ldots\ldots\ldots\right\}\ \ldots\quad(1)$$

Now

$$\int_0^\infty \frac{2\sin at}{\cosh \pi t}.\frac{\sin(4n+2)xt}{2\cos xt}\,dt$$

$$=\int_0^\infty\left\{\frac{\sin\frac{at}{2x}}{\cosh\frac{\pi t}{2x}}.\frac{\sin(2n+1)t}{2\cos\frac{t}{2}}\frac{dt}{x}\right\},\text{ by changing } 2xt \text{ into } t$$

$$=\int_0^\infty\left\{\frac{\sin\frac{at}{2x}.\sin\frac{t}{2}}{\cosh\frac{\pi t}{2x}}.\frac{\sin(2n+1)t}{\sin t}\frac{dt}{x}.\right\}$$

Let n become infinite; then the value of the above integral is

$$\frac{\pi}{x}\left\{\frac{\sin\frac{a\pi}{2x}}{\cosh\frac{\pi^2}{2x}}-\frac{\sin\frac{3a\pi}{2x}}{\cosh\frac{3\pi^2}{2x}}+\frac{\sin\frac{5a\pi}{2x}}{\frac{5\pi^2}{2x}}\ldots\ldots\right\}$$

$$\tfrac{1}{2}\left\{\text{sech}\,\frac{x-a}{2}-\text{sech}\,\frac{3x-a}{2}+\text{sech}\,\frac{5x-a}{2}\text{.........to }\infty\right\}$$

$$=e^{-\frac{1}{2}(x-a)}-e^{-\frac{3}{2}(x-a)}+e^{-\frac{5}{2}(x-a)}-\ldots\ldots\ldots$$

$$-e^{-\frac{1}{2}(3x-a)}+e^{-\frac{3}{2}(3x-a)}-e^{-\frac{5}{2}(3x-a)}+\ldots\ldots\ldots$$

$$\begin{matrix}\cdots & \cdots & \cdots & \cdots & \cdots\\ \cdots & \cdots & \cdots & \cdots & \cdots\end{matrix}$$

This double series is absolutely convergent if $x>a>o$. Summing by columns, we get

$$\frac{1}{2}\left\{\frac{e^{\frac{a}{2}}}{\cosh\frac{x}{2}}-\frac{e^{\frac{3a}{2}}}{\cosh\frac{3x}{2}}+\frac{e^{\frac{5a}{2}}}{\cosh\frac{5x}{2}}\ldots\ldots\ldots\ldots\right\}$$

Similarly,

$$\frac{1}{2}\left\{\text{sech}\,\frac{x+a}{2}-\text{sech}\,\frac{3x+a}{2}+\text{sech}\,\frac{5x+a}{2}\ldots\ldots\ldots\right.$$

$$=\frac{1}{2}\left\{\frac{e^{-\frac{a}{2}}}{\cosh\frac{x}{2}}-\frac{e^{-\frac{3a}{2}}}{\cosh\frac{3x}{2}}+\frac{e^{-\frac{5a}{2}}}{\cosh\frac{5x}{2}}\ldots\ldots\ldots\ldots\right\}$$

$\therefore$ Thus the right hand side of (1) when n is infinite becomes

$$\frac{\sinh\frac{a}{2}}{\cosh\frac{x}{2}}-\frac{\sinh\frac{3a}{2}}{\cosh\frac{3x}{2}}+\frac{\sinh\frac{5a}{2}}{\cosh\frac{5x}{2}}\ldots\ldots\ldots .$$

Hence $$\sum_{n=0}^{\infty}(-1)^n\frac{\pi}{x}\sin\frac{a\pi(2n+1)}{2x}.\ \text{sech}\,\frac{\pi^2(2n+1)}{2x}$$

$$=\sum_{n=0}^{\infty}(-1)^n\sinh\frac{a(2n+1)}{2}.\ \text{sech}\,\frac{x(2n+1)}{2}\qquad\ldots\quad(2)$$

Equationg the coefficients of a^{2m+1}, we get

$$\sum_{n=0}^{\infty}(-1)^n\left[(2n+1)^{2m+1}\left\{\text{sech}\,\frac{x(2n+1)}{2}-\right.\right.$$

$$\left.\left.(-1)^m\left(\frac{\pi}{x}\right)^{2m+2}\text{sech}\,\frac{\pi^2(2n+1)}{2x}\right\}\right]=0.\qquad\ldots\quad(3)$$

Putting $x=\pi$, we have

$$\sum_{n=0}^{\infty}(-1)^n\left[(2n+1)^{2m+1}\left\{\text{sech}\,\frac{\pi(2n+1)}{2}\left(1-(-1)^m\right)\right\}\right]=0.$$

When m is even we get no effective result, as the left hand side vanishes. When $m=2r-1$, $(r\neq 0)$

$$\sum_{n=0}^{\infty}(-1)^n(2n+1)^{4r-1}\operatorname{sech}\frac{\pi(2n+1)}{2}=0,$$

which is the required result with r instead of n.

When r is zero, the series becomes

$$\operatorname{sech}\frac{\pi}{2}-\operatorname{sech}\frac{3\pi}{2}+\frac{1}{5}\operatorname{sech}\frac{5\pi}{2}\ldots\ldots\ldots\ldots\ldots$$

The value of this series is $\frac{\pi}{8}$ and is got by making $n=1$ in Mr. S. Ramanujan's series of Question 353 or by Question 626 (vi).

Question 418.

(M. Bhimasena Rao) :—Tangents at P, Q, R, S on a cubic curve having a centre, co-intersect on the curve. Show that the mean centre of P, Q, R, S, is the centre of the curve, each asymptote bisecting a pair of opposite connectors of the quadrangle P Q R S.

Solution by N. Sankara Aiyar, M.A.

Taking the origin at the centre and two of the asymptotes as axes, since the curve has a centre, its equation can be reduced to the form

$$(lx+my)xy+ax+by=0.$$

The tangent at (f, g) on this is

$$(2lgf+mg^2+a)\xi+(2mgf+lf^2+b)\eta+2af+2bg=0.$$

If the point (ξ, η) be common to this tangent and the curve, by eliminating g between this equation and $(lf+mg)fg+af+bg=0$, we shall get an equation giving values of f for which the tangents will be concurrent at (ξ, η).

Now $$g^2m\xi+g(2lf\xi+2mf\eta+2b)+a\xi+lf^2\eta+b\eta+2af=0$$

and $$g^2fm+g(lf^2+b)+af=0.$$

$$\therefore \{fm(a\xi+lf^2\eta+b\eta+2af)-afm\xi\}^2$$
$$=\{mf(lf^2+b)-fm(2lf\xi+2mf\eta+2b)\}\times$$
$$\{af(2lf\xi+2mf\eta+2b)-lf^2+b)(a\xi+lf^2\eta+b\eta+2af)\}$$

Evidently two of the roots of this sextic are f.

To find the equation giving the four required roots, we divide this by $f^2-2\xi f+\xi^2$.

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Now the coefficient of f^6 is

$$l^3\xi\eta+2ml^2\eta^2-ml^2\eta^2=l^2\eta(l\xi+m\eta).$$

The coefficient of f^5 is

$$(2l^2\eta b+2al^2\xi-2alm\eta+2alm\eta)=2l^2(a\xi+b\eta)$$

$$=-2l^2\xi\eta(l\xi+m\eta), \text{ by the equation of curve.}$$

The coefficient of f is

$$2b^2(a\xi+b\eta)=-2b^2\xi\eta(l\xi+m\eta)$$

The absolute term is

$$-b^2\xi(a\xi+b\eta)=b^2\xi^2\eta(l\xi+m\eta)$$

Hence the equation reduces to

$$(f^2-2\xi f+\xi^2)\left(f^4+pf^2+\frac{b^2}{l^2}\right)=0.$$

i.e. $f^4+pf^2+\frac{b^2}{l^2}=0$ is the equation giving P, Q, R, S.

i.e. $(f^2-\alpha^2)(f^2-\beta^2)=0.$

i.e. $f=\pm\alpha, \pm\beta.$

Hence the ξ—co-ordinate of the mean centre of P, Q, R, S; is $\frac{\alpha-\alpha+\beta-\beta}{4}=0$. Similarly the η—co-ordinate is 0. Hence the centre is the mean centre of P, Q, R, S.

Again, the middle point of PQ will have co-ordinates $\frac{\alpha-\alpha}{2}=0.$ Hence this point lies on one of the asymptotes; similarly that of R S lies on this. Similarly the other two pairs of opposite connectors are bisected by the other two asymptotes.

Question 468.

(N. Sankara Aiyar, M.A.):—Solve the equation

$$\int_0^{v_0}\left\{\frac{x^2\left(\frac{dy}{dx}\right)^2+1}{v_0^2-x^2}\right\}^{\frac{1}{2}}dx=k.$$

Solution by A. Narasinga Rao.

Put $x=v_0\cos\theta$ ... ... (1)

The integral equation reduces to

$$\int_0^{\frac{\pi}{2}}\left\{\cot^2\theta\left(\frac{dy}{d\theta}\right)^2+1\right\}^{\frac{1}{2}}d\theta=k;$$

i.e.
$$\int_{0}^{\frac{\pi}{2}} f(\theta)d\theta=k,$$

i.e.
$$F\left(\frac{\pi}{2}\right)-F(o)=k.$$

where $F(\theta)=\int f(\theta)d\theta$ ... ... (2)

If any solution F of (2) be obtained, we have a corresponding f and y is given in terms of θ by the equation.

$$\cot^2\theta\left(\frac{dy}{d\theta}\right)^2+1=\{f(\theta)\}^2.$$

The elimination of θ between this value of y and (1) determines y as a function of x.

Put $F(\theta)=k\sin\theta+\theta\left(\theta-\frac{\pi}{2}\right)G(\theta)+\alpha$, α being an arbitrary constant. This value of F will satisfy (2) if G is an one-valued function of θ, finite and continuous in the range $0\leq\theta\leq\frac{\pi}{2}$.

For instance, take the simplest case $\alpha=0$, $G(\theta)=0$.

The relation determining y is now

$$\cot^2\theta\left(\frac{dy}{d\theta}\right)^2+1=k^2\cos^2\theta,$$

i.e.
$$x^2\left(\frac{dy}{dx}\right)^2+1=k^2\frac{x^2}{v_o^2}\ \text{from}(1)$$

Hence
$$y=\int\left(\frac{k^2}{v_o^2}e^{2z}-1\right)^{\frac{1}{2}}dz,$$

where $z=\log x$.

Other forms of F may be obtained by inspection from equation (2). For instance

(i) $F(\theta)=\Sigma(A_r\sin^r\theta+B_r\cos^r\theta)$ the coefficients being connected by the relation $\sum_{r=1}(A_r-B_r)=k$

(ii) $F(\theta)=a^{2\theta/\pi}G(\sin\overline{4\theta+c})$ where G is one-valued and a, c are connected by $(a-1)G(\sin c)=k$

Questions 489, 608, 626.

Q. 489. (S. Ramanujan) :—Shew that

$$(1+e^{-\pi\sqrt{55}})(1+e^{-3\pi\sqrt{55}})(1+e^{-5\pi\sqrt{55}})\ldots\ldots$$
$$=\frac{1+\sqrt{(3+2\sqrt{5})}}{\sqrt{2}}\left\{e^{-\pi\sqrt{55}}\right\}^{\frac{1}{24}}$$

Q. 608. (M. BHIMASENA RAO) :—Shew that

$$\text{(i)}\quad (1+e^{-\pi})(1+e^{-3\pi})(1+e^{-5\pi})\ldots=2^{\frac{1}{4}}e^{-\frac{\pi}{24}}$$

$$\text{(ii)}\quad (1+e^{-\pi\sqrt{3}})(1+e^{-3\pi\sqrt{3}})\ldots=2^{\frac{1}{3}}\cdot e^{-\frac{\pi\sqrt{3}}{24}}$$

$$\text{(iii)}\quad (1+e^{-\pi\sqrt{7}})(1+e^{-3\pi\sqrt{7}})\ldots=2^{\frac{1}{2}}\cdot e^{-\frac{\pi\sqrt{7}}{24}}.$$

Q. 626. (M. BHIMASENA RAO) :—If $f(x)$ denotes

$$\operatorname{sech}\frac{\pi x}{2}-\frac{1}{3}\operatorname{sech}\frac{3\pi x}{2}+\frac{1}{5}\operatorname{sech}\frac{5\pi x}{2}-\ldots$$

shew that

$$\text{(i)}\quad f(1)=\frac{\pi}{8}$$

$$\text{(ii)}\quad f(\sqrt{3})=\frac{\pi}{24}$$

$$\text{(iii)}\quad f(\sqrt{7})=\frac{1}{2}\left[\sin^{-1}\left(\frac{3}{4}\right)-\frac{\pi}{4}\right]$$

$$\text{(iv)}\quad f(\sqrt{15})=\frac{1}{2}\left(\frac{\pi}{12}-\sin^{-1}\frac{1}{4}\right)$$

$$\text{(v)}\quad f\left(\sqrt{\frac{3}{5}}\right)=\frac{1}{2}\left(\cos^{-1}\frac{1}{4}-\frac{\pi}{12}\right)$$

$$\text{(vi)}\quad f(x)+f(x^{-1})=\frac{\pi}{4}.$$

Are there any values of x other than zero and those given by (vi) for which $f(x)$ can be evalatted in finite terms?

Solution by A. C. L. Wilkinson, M.A., F.R.A.S.

Weber, Lehr buct der Algebra, Vol III, P. 86 **(11)** gives

$$q^{-\frac{1}{24}}\prod_{1}^{\infty}(1+q^{2n-1})=f(w),\quad \text{where } q=e^{i\pi w}$$

Also in the table of class invariants

$$f(\sqrt{-55})=\sqrt{2}x,\; x^2-x=\frac{1+\sqrt{5}}{2},$$

whence Mr. Ramanujan's result follows.

Again $f(\sqrt{-1})=\sqrt[4]{2}$, $f(\sqrt{-3})=\sqrt[3]{2}$, $f(\sqrt{-7})=\sqrt{2}$, whence the results of Question 608.

Jacobi, Fundamenta Nova, p. 108 (46) gives the formula

$$\frac{1}{2}\sin^{-1}k=\frac{2}{q^{1/2}+q^{-\frac{1}{2}}}-\frac{1}{3}\cdot\frac{2}{q^{3/2}+q^{-\frac{3}{2}}}+\frac{1}{5}\cdot\frac{2}{q^{5/2}+q^{-\frac{5}{2}}}-\ldots$$

Put $$q=e^{-\pi x}=e^{-\pi k'/k}$$

(1) For $x=1$, $k=k'$, $k=k'=\frac{1}{\sqrt{2}}$ etc.

(2) For $x=\sqrt{3}$, $2kk'=\sin\frac{\pi}{6}$; vide: Greenhill; *Elliptic Functions*, p. 331.

(3) For $x=\sqrt{7}$, $2kk'=\frac{1}{8}$, Greenhill p. 331.

(4) For $x=\sqrt{15}$, $\sqrt[4]{kk'}=\sin 18$; Greenhill has $\sqrt{kk'}=\sin 18$, an obvious misprint.

(6) $f(x)+f(x^{-1})=\frac{1}{2}(\sin^{-1}k+\sin^{-1}k')=\frac{\pi}{4}$.

Question 530.

(N. Sankara Aiyar, M.A.) :—Solve in integers

$$\frac{x^2+17}{y+9}=\text{a perfect square.}$$

Solution by A. Narasinga Rao.

Let $$\frac{x^2+17}{y+9}=z^2 \quad \dots \quad \dots \quad \dots (1)$$

Put $y+9=y_1$ so that $x^2+17=y_1 z^2$.

Give x any integral value n, and write (n^2+17) in terms of its prime factors. Say

$$(n^2+17)=p_1^{2\alpha_1+\beta_1} p_2^{2\alpha_2+\beta_2} \quad \dots \quad \dots \quad \dots (2)$$

where $p_1, p_2 \dots$ are the prime factors of (n^2+17) and the β's are either 0 or 1.

It is obvious from (2) that we may take for z any one of the terms in the distributed product $(1+p_1+p_1^2+\dots p_1^{\alpha_1})(1+p_2+p_2^2+\dots p_2^{\alpha_2})\dots$ If any of these values of z be substituted in (2), we obtain the corresponding value of y_1 and thence that of y. The aggregate of all these sets of values for all integral values of x constitute the complete solution.

A simple solution is

$$z=1,\ x=n,\ y=n^2+8.$$

The question may also be solved by giving values to y_1, and solving the equation

$$x^2-y_1z^2=-17,$$

by the usual methods.

Question 533.

(PROF. K. J. SANJANA):—Three real numbers a, b, c, are written down at random. Find the chance (i) that $b^2 < 4ac$, (ii) that $b^2 > 4ac$, what is the chance of b^2 being equal to $4ac$?

Solution by N. Sankara Aiyar, M. A.

Case (i) has already been solved on p. 67, Vol. VI, of J. I. M. S.

Case (ii). If a and c have opposite signs, then $b^2 > 4ac$. The chance of this is $\frac{1}{2}$.

If a & c have the same sign, if a is greatest the condition reduces to $b^2 > 4c$; if c is greatest, to $b^2 > 4a$; and if b is greatest, to $1 > 4ac$ where a, b, c are $\angle 1$. The chance of each of these cases is $\frac{1}{3}$. In case a is greatest, c ranges from $\frac{b^2}{4}$ to 0 and b ranges from 1 to 0. Hence the chance of $b^2 > 4c$ in this case is

$$\int_0^1 db \int_0^{\frac{b^2}{4}} dc = \left[\frac{b^3}{12}\right]_0^1 = \frac{1}{12}.$$

Similarly $\frac{1}{12}$ is the chance when c is greatest.

When b is greatest, if $c < \frac{1}{4}$, $4ac < 1$. The chance of this is clearly $\frac{1}{4}$. When $c > \frac{1}{4}$, a ranges from $\frac{1}{4c}$ to 0 and c from 1 to $\frac{1}{4}$. The chance of $1 > 4ac$ is

$$\int_{\frac{1}{4}}^1 dc \int_0^{\frac{1}{4c}} da = \left[\frac{1}{4}\log c\right]_{\frac{1}{4}}^1 = \frac{1}{2}\log 2.$$

Hence the total chance of b^2 being greater than $4ac$ is

$$\frac{1}{2} + \frac{1}{2} \cdot \frac{1}{3}\left(\frac{1}{12} + \frac{1}{12} + \frac{1}{4} + \frac{1}{2}\log 2\right)$$

$$= \frac{1}{2} + \frac{5}{72} + \frac{1}{12}\log 2 = \frac{41}{72} + \frac{1}{12}\log 2.$$

Hence the chance of $b^2 = 4ac$ is zero. This is explained by the fact that for any given value of b^2 and of a there is only one out of an infinity of values which will give $b^2 = 4ac$, and so the chance is $\frac{1}{\infty}$; the chance of $b^2 = 4ac$ for any given value of b is therefore $\frac{1}{2} \cdot \frac{1}{\infty}$ since a & c should have the same sign. That is, the chance of $b^2 = 4ac$ is $\frac{1}{2}\frac{1}{\infty}$ or zero *i.e.*, negligible.

Question 545.

(M. T. Naraniengar, M.A.) :—Show that the roots of $8x^3-18x+9=0$ and $8y^3-18y-9=0$ are connected by the relations

$$x=3+2y-2y^2 \text{ and } y=3-x-2x^2.$$

Solution by N. Sankara Aiyar, M.A.

We have
$$8x^3-18x+9=0$$
$$8y^3-18y-9=0.$$

Adding, $$8(x^3+y^3)-18(x+y)=0.$$

$\therefore$ $$4(x^2-xy+y^2)-9=0; \text{ since } x+y\neq 0.$$

i.e. $$4x^2-4xy+4y^2-9=0.$$

i.e. $$(2x-y)=\pm\sqrt{9-3y^2}\ldots\ldots(1)$$

Now
$$(2y-3)(8y^3-18y-9)$$
$$=16y^4-24y^3-36y^2+36y+27=0$$

$\therefore$ $$16y^4-24y^3-39y^2+36y+36=(9-3y^2)$$

i.e. $$(4y^2-3y-6)^2=(9-3y^2).$$

$\therefore$ $$2x-y=4y^2-3y-6 \text{ or } 6+3y-4y^2 \text{ by (1)}$$

i.e. $$x=2y^2-y-3 \text{ or } 3+2y-2y^2.$$

Similarly, $(2y-x)=\pm\sqrt{9-3x^2}$

And
$$(2x+3)(8x^3-18x+9)$$
$$=16x^4+24x^3-36x^2-36x+27$$
$$=16x^4+24x^3-39x^2-36x+36-(9-3x^2)$$
$$=(4x^2+3x-6)^2-(9-3x^2).$$

$\therefore$ $$2y-x=\pm(6-3x-4x^2)$$

i.e. $$y=3-x-2x^2 \text{ or } 2x^2+2x-3.$$

Question 546.

(S. Ramanujan) :—Shew that

(i) $$\left(\frac{1}{3}-\frac{1}{4}\right)+\frac{2}{3^2}\left(\frac{1}{3}-\frac{1}{4^2}\right)+\frac{2\cdot4}{3\cdot5^2}\left(\frac{1}{3}-\frac{1}{4^3}\right)+=\frac{\pi}{12}\log(2+\sqrt{3});$$

(ii) $$1-\frac{2}{3^2}+\frac{2\cdot4}{3\cdot5^2}-\frac{2\cdot4\cdot6}{3\cdot5\cdot7^2}+\ldots=\frac{\pi^2}{8}-\frac{1}{2}\{\log(1+\sqrt{2})\}^2.$$

Solution by M. Bhimasena Rao.

(i) $$\left(\frac{1}{3}-\frac{1}{4}\right)+\frac{2}{3^2}\left(\frac{1}{3}-\frac{1}{4^2}\right)-\frac{2.4}{3.5^2}\left(\frac{1}{3}-\frac{1}{5^2}\right)+\ldots$$

$$=\frac{1}{3}\left(1+\frac{2}{3^2}+\frac{2.4}{3.5^2}+\ldots\right)-\frac{1}{2}\left\{\frac{1}{2}+\frac{2}{3^2}\cdot\frac{1}{2^3}+\frac{2.4}{3.5^2}\cdot\frac{1}{2^5}+\ldots\right\}$$

$$=\frac{1}{3}\int_0^1 \frac{\sin^{-1}x}{\sqrt{1-x^2}}\cdot\frac{dx}{x}-\frac{1}{2}\int_0^{\frac{1}{2}}\frac{\sin^{-1}x}{\sqrt{1-x^2}}\frac{dx}{x}.$$

Now

$$\int_0^x \frac{\sin^{-1}x}{\sqrt{1-x^2}}\cdot\frac{dx}{x}=\sin^{-1}x\log\frac{x}{1+\sqrt{1-x^2}}-\int_0^x \log\frac{x}{1-\sqrt{1-x^2}}\,d\sin^{-1}x$$

$$=\sin^{-1}x\log\frac{x}{1+\sqrt{1-x^2}}-\int_0^{\sin^{-1}x}\log\frac{\sin\theta}{1+\cos\theta}\,d\theta,\ \text{put } \sin\theta=x$$

$$=\sin^{-1}x\log\frac{x}{1+\sqrt{1-x^2}}+\int_0^{\sin^{-1}x}\log\cot\frac{\theta}{2}.\,d\theta$$

$$=\sin^{-1}x\log\frac{x}{1+\sqrt{1-x^2}}+\int_0^{\sin^{-1}x}2\left(\cos\theta+\frac{\cos 3\theta}{3}+\frac{\cos 5\theta}{5}+\ldots\right)d\theta$$

$$=\sin^{-1}x\log\frac{x}{1+\sqrt{1-x^2}}+2\left(\sin\theta+\frac{\sin 3\theta}{3^2}+\frac{\sin 5\theta}{5^2}+\ldots\ldots\right)$$

where $\sin\theta=x$.

$$\therefore\quad \int_0^1 \frac{\sin^{-1}x}{\sqrt{1-x^2}}\cdot\frac{dx}{x}=2\left(1-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\ldots\ldots\right)$$

and

$$\int_0^{\frac{1}{2}}\frac{\sin^{-1}x}{\sqrt{1-x^2}}\cdot\frac{dx}{x}=\frac{\pi}{6}\log\frac{1}{2+\sqrt{3}}+2\left(\sin\frac{\pi}{6}+\frac{\sin\frac{3\pi}{6}}{3^2}+\frac{\sin\frac{5\pi}{6}}{5^2}+\ldots\right)$$

$$=\frac{\pi}{6}\log\frac{1}{2+\sqrt{3}}+2\left(\frac{1}{2}+\frac{1}{3^2}+\frac{1}{2}\cdot\frac{1}{5^2}-\frac{1}{2}\cdot\frac{1}{7^2}-\frac{1}{9^2}-\frac{1}{2}\cdot\frac{1}{11^2}\cdots\right)$$

$$=\frac{\pi}{6}\log\frac{1}{2+\sqrt{3}}+2\left\{\frac{1}{2}\left(1-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\frac{1}{9^2}-\frac{1}{11^2}\cdots\right)+\frac{1}{2}\left(\frac{1}{3^2}-\frac{1}{9^2}+\frac{1}{15^2}-\cdots\right)+\left(\frac{1}{3^2}-\frac{1}{9^2}+\frac{1}{15^2}\ldots\ldots\right)\right\}$$

$$=\frac{\pi}{6}\log\frac{1}{2+\sqrt{3}}+\frac{4}{3}\left(1-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\ldots\ldots\right)$$

$$\therefore\ \frac{1}{3}\int_0^1\frac{\sin^{-1}x\,dx}{x\sqrt{1-x^2}}-\frac{1}{2}\int_0^{\frac{1}{2}}\frac{\sin^{-1}x\,dx}{x\sqrt{1-x^2}}$$

$$=\frac{2}{3}\left(1-\frac{1}{3^2}+\frac{1}{5^2}-\ldots\right)-\frac{\pi}{12}\log\frac{1}{2+\sqrt{3}}-\frac{2}{3}\left(1-\frac{1}{3^2}+\frac{1}{5^2}-\ldots\right)$$

$$=\frac{\pi}{12}\log(2+\sqrt{3})$$

(ii) Consider the series

$$x-\frac{2}{3^2}.x^3+\frac{2\cdot 4}{3\cdot 5^2}.x^5-\ldots\ldots$$

This is equal to

$$\int_0^{\pi}\frac{\sinh^{-1}x}{x\sqrt{1+x^2}}dx$$

$$=\int_0^{\sinh^{-1}x}\frac{ydy}{\sinh y}=\int_0^{\sinh^{-1}x} yd\log\tanh\frac{y}{2}$$

$$=2\int_0^{\tanh\frac{\sinh^{-1}x}{2}}\frac{\tanh^{-1}z}{z}dz=2\int_0^{\tanh\frac{\sinh^{-1}x}{2}}\sum\frac{z^{2n+1}}{(2n+1)^2}$$

$$\equiv 2\,f(z)\text{ where } z=\frac{\sqrt{1+x^2}-1}{x}$$

It may be shown that

$$f(z)+f\left(\frac{1-z}{1+z}\right)=\frac{\pi^2}{8}-\tfrac{1}{2}\log z.\log\frac{1-z}{1+z}\ldots \qquad \ldots (1)$$

Putting $z=\frac{1-z}{1+z}$, we get $z=\sqrt{2}\pm 1$,

Taking the value $\sqrt{2}-1$, $x=1$ and by (1)

$$f(\sqrt{2}-1)=\frac{\pi^2}{16}-\tfrac{1}{4}\{\log(\sqrt{2}-1)\}^2$$

$$=\frac{\pi^2}{16}-\tfrac{1}{4}\{\log(1+\sqrt{2}\}^2.$$

$$\therefore\ 1-\frac{2}{3^2}+\frac{2\cdot 4}{3\cdot 5^2}=\ldots=2\,f(\sqrt{2}-1)$$

$$=\frac{\pi^2}{8}-\tfrac{1}{2}\{\log(1+\sqrt{2})\}^2.$$

Mr. N. Sankara Aiyar sends a solution of (ii) *only on similar lines.*

Question 575.

(S. NARAYANA AIYAR, M.A.):—If B_1, B_3, B_5, B_7, &c., are the Bernoulle's numbers $\frac{1}{6}$, $\frac{1}{30}$, $\frac{1}{42}$, $\frac{1}{30}$, &c., show that

$$B_1-B_3+B_5-B_7+B_9-B_{11}+\ldots\text{ to }\infty=\frac{\pi^2-9}{6}$$

Solution (1) *by R. S. Pocock, B.A., B.Sc., F.R.A.S., and* (2) *by K. B. Madhava.*

(1) We have the usual formula in the calculus of finite differences

viz. $$\sum_{1}^{x-1} u_x = C + \int u_x dx - \frac{1}{2}u_x + \frac{B_1}{2!}\cdot\frac{du_x}{dx} - \frac{B_3}{4!}\cdot\frac{d^3u_x}{d_x^3} + \dots$$

Writing $u_x = \frac{1}{x^2}$ and adding $\frac{1}{x^2}$ to each side,

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\dots\frac{1}{x^2} = C - \frac{1}{x} - \frac{1}{2x^2} + \frac{1}{x^2} - \frac{B_1}{x^3} + \frac{B_3}{x^5} - \frac{B_5}{x^7} + \dots$$

Putting $x=\infty$, we find $C=\frac{\pi^2}{6}$; and putting $x=1$, we get

$$B_1 - B_3 + B_3 - B_5 + \dots = \frac{\pi^2-9}{6}.$$

(2) This is at once deduced from formula (29) obtained by Mr. S. Ramanujan on p. 224 of J. I. M. S., Dec. 1911 (Vol. 3) which states except for notation about the suffixes of B

$$\frac{1}{(n+1)^2}+\frac{1}{(n+2)^2}+\frac{1}{(n+3)^2}+\dots\infty = \frac{1}{n} - \frac{1}{2n^2} + \frac{B_1}{n^3} - \frac{B_3}{n^5} + \frac{B_5}{n^7}\dots\infty$$

With $n=1$ and the known value for $\sum_{1}^{\infty}\frac{1}{n^2}$ *viz.* $\frac{\pi^2}{6}$, we have at once

$$B_1 - B_3 + B_5 \dots = \frac{\pi^2-9}{6}.$$

This is also obtained from Ex: 26, p. 489 of Bromwich *viz.*,

$$\frac{1}{1+x}+\frac{1}{2(1+2x)}+\frac{1}{3(1+3x)}+\dots = C + \log\frac{1}{x} + \frac{x}{2} - B_1\frac{x^2}{2} + B_3\frac{x^4}{4}\dots$$

By differentiation; which is exactly identical with Mr. Ramanujan's; except that this result can be established by applying Euler's formula (§131, Bromwich) to the function $f=\frac{1}{t(1+xt)}$.

Question 576.

(S. Narayana Aiyar, M. A.) :—Shew that

$$e^x = \frac{1+x\frac{a}{b}+\frac{x^2}{1\cdot 2}\frac{a(a+1)}{b(b+1)}+\dots\dots\text{to }\infty}{1+x\frac{a-b}{b}+\frac{x^2}{1\cdot 2}\frac{(a-b)(a-b-1)}{b(b+1)}+\dots\dots\text{to }\infty}$$

Solution (1) *by T. P. Trivedi, M.A., LL.B. and*
(2) *by M. Bhimasena Rao.*

(1) $1+x.\,\frac{a}{b}+\frac{x^2}{1\cdot 2}\,\frac{a(a+1)}{b(b+1)}+\dots\dots$to ∞ is the coefficient of z^{a-1} in

$$\underline{|b-1}\left\{(z+x)^{a-b} + \frac{(z+x)^{a-b+1}}{\underline{|1}} + \dots\dots\frac{(z+x)^{a-1}}{\underline{|b-1}}\right.$$

$$+\frac{(z+x)^a}{\underline{|b}}+\frac{(z+x)^{a+1}}{\underline{|b+1}}+\ldots\ldots\ldots\ldots\ldots\text{to } \infty\Big\}$$

i.e. in $$\underline{|b-1}(z+x)^{a-b}\left\{1+\frac{z+x}{\underline{|1}}+\frac{(z+x)^2}{\underline{|2}}+\text{ad in } f.\right\}$$

i.e. in $$\underline{|b-1}(z+x)^{a-b}e^{x+z}$$

$$=\underline{|b-1}e^x\times\text{coeff of } z^{a-1}\text{ in}(z+x)^{a-b}e^z=\underline{|b-1}e^x\times\text{coeff of } z^{a-1}\text{ in}$$

$$\left\{z^{a-b}+{}^{(a-b)}c_1z^{a-b-1}x+{}^{(a-b)}c_2x^{a-b-2}+\ldots\ldots\right\}\times$$

$$\left\{1+z+\frac{z^2}{\underline{|2}}+\ldots\ldots\frac{z^{b-1}}{\underline{|b-1}}+\frac{x^b}{\underline{|b}}+\frac{z^{b+1}}{\underline{|b+1}}+\ldots\ldots\ldots\right\}$$

$$=\underline{|b-1}e^x\left\{\frac{1}{\underline{|b-1}}+{}^{(a-b)}c_1\,x.\frac{1}{\underline{|b}}+{}^{(a-b)}c_2\,x^2\frac{1}{\underline{|b-2}}+\ldots\ldots\ldots\right\}$$

$$=e^x\left\{1+x.\frac{a-b}{b}+\frac{x^2}{1\cdot 2}\frac{(a-b)(a-b-1)}{b(b+1)}+\ldots\ldots\ldots\right\}$$

whence the result

If $a-b$ be an integer, the series in the *den*r will be finite, but this restriction is not necessary for the question.

Quest. 616 suggests another method for obtaining the result. It may also be obtained by multiplying the denominator by the series for e^x and identifying the coefficients.

(2) If $b>a>o$, we have

$$\frac{a(a+1)(a+2)\ldots\ldots(a+n-1)}{b(b+1)(b+2)\ldots\ldots(b+n-1)}=\frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\int_o^1 t^{a+n-1}(1-t)^{b-a-1}dt.$$

$$\therefore\ 1+x\frac{a}{b}+\frac{x^2}{2\,!}\frac{a(a+1)}{b(b+1)}+\ldots\ldots\ldots$$

$$=\frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\int_o^1\Big\{t^{a-1}(1-t)^{b-a-1}+t^a(1-t)^{b-a-1}x+$$

$$t^{a+1}(1-t)^{b-a-1}\frac{x^2}{2\,!}+\ldots\Big\}dt$$

$$=\frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\int_o^1 t^{a-1}(1-t)^{b-a-1}e^{xt}dt$$

$$=\frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\int_o^1 (1-t)^{a-1}t^{b-a-1}e^{x(1-t)}\,dt$$

$$=e^x.\frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\int_o^1 t^{(b-a)-1}(1-t)^{a-1}e^{-xt}\,dt$$

$$=e^x\left\{1-\frac{(b-a)}{b}x+\frac{(b-a)(b-a+1)}{b(b+1)}\frac{x^2}{2\,!}-\ldots\ldots\right\},$$

by expanding e^{-xt} and integrating

$$\text{Hence } e^x=\frac{1+x\frac{a}{b}+\frac{x^2}{2\,!}\frac{a(a+1)}{b(b+1)}+\frac{x^3}{3\,!}\frac{a(a+1)(v+2)}{b(b+1)(b+2)}+\ldots\ldots}{1+\frac{a-b}{b}x+\frac{(a-b)(a-b-1)}{b(b+1)}\frac{x^2}{2\,!}+\ldots\ldots}$$

Question 579.

(S. Narayanan, B. A., L. T.):—With the three vertices of a ΔABC as centres circles are described to touch the opposite sides. Shew that the common chords of these circles with the circumcircle form a triangle which is in perspective with the original triangle.

Solution (1) *by V. V. S. Narayan* ; (2) *by D. Krishnamurti and N. Sankara Aiyar.*

(1) Let $\alpha\beta\gamma$ be the Δ formed by the common chords, and let the corresponding sides of $\alpha\beta\gamma$ and ABC intersect at P, Q, R. We shall prove that P, Q, R are collinear. In the figure AD is $\perp$ to BC and AM = AN = AD, by hypothesis.

$\therefore$ $\quad A\hat{M}N = A\hat{N}M = A\hat{B}M.$

$\therefore$ $\quad AX.\,AB = AM^2 = AD^2.$

$\therefore$ $\quad$ DX is $\perp$ to AB.

Similarly $\quad$ DY is $\perp$ to AC.

Thus $\quad BX/XA = DB^2/DA^2$, from similar triangles.

and $\quad AY/YC = DA^2/CD^2.$

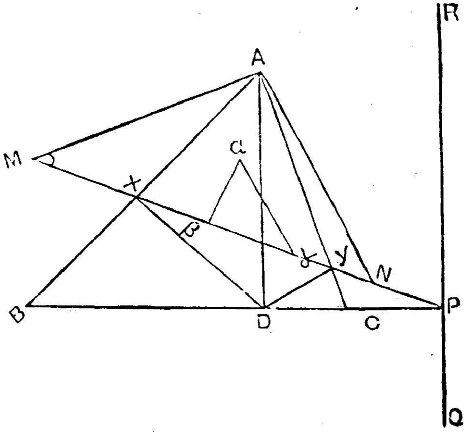
$\therefore$ $\quad BX/XA.\,AY/YC = DB^2/CD^2.$

But $\quad BX/XA.\,AY/YC.CP/PB = 1$, by Menelau's Theorem.

$\therefore$ $\quad CP/PB = CD^2/DB^2.$

If DEF is the pedal Δ, we find similarly,

$$BR/RA = BF^2/FA^2,$$
$$AQ/QC = AE^2/EC^2.$$



Compounding,

$$CP/PB.\ BR/RA.\ AQ/QC = CD^2/DB^2.\ BF^2/FA^2.\ AE^2/EC^2 = 1,$$
since AD, BE, CF are concurrent.

Hence P, Q, R, are collinear. That is, the Δ^s ABC, $\alpha\beta\gamma$ are coaxial. But coaxial Δ^s are also copolar. Hence Δ^sABC, $\alpha\beta\gamma$ are copolar.

(2) The equation in areal co-ordinates to a circle of radius r can be written in the form

$$f^2x+g^2y+h^2z=r^2+S,$$
where fgh are the distances of the centre from A,B,C, and S = 0 represents the equation to the circum-circle: [*Vide* Askwith's *Analytical Geometry*, p. 313, Q. 22.]

The equation to the circle touching BC and having A for centre is thus
$$c^2y+b^2z=p^2_1+S,$$
p_1 being the $\perp^r$ from A on BC.

The equation to the common chord of this and the circumcircle is
$$c^2y+b^2z=p^2_1(x+y+z) \quad \dots \quad \dots (1)$$

The co-ordinates of the point L of intersection of (1) with BC, or $x=0$ are proportional to

$$x:y:z=0:-(b^2-p_1^2):c^2-p_1^2$$
$$=0:-b^2(1-c^2\lambda^2):c^2(1-b^2\lambda^2)$$
$$=0:-C'b^2:B'c^2,$$
where $\sin A/a = \dots = \lambda$, and $A'=(1-a^2\lambda^2)$, etc.

Similarly the other points M and N can be got.

Since the determinant formed by the co-ordinates of L, M, N, *viz.*,

$$\begin{vmatrix} 0 & -C'b^2 & B'c^2 \\ Ca^2 & 0 & A'c^2 \\ -Ba^2 & Ab^2 & 0 \end{vmatrix}$$
is identically zero, L, M, N are collinear and the result follows.

Question 582.

(J. C. Swaminarayan, M. A.) :—Prove the identities

$$\tan^{-1} x - \tan^{-1}\frac{x}{3} + \tan^{-1}\frac{x}{5} - \tan^{-1}\frac{x}{7} + \dots\dots = \tfrac{1}{2}\,\mathrm{gd}\frac{\pi}{2}x$$

and
$$\tanh^{-1} x - \tanh^{-1}\frac{x}{3} + \tanh^{-1}\frac{x}{5} \dots\dots = \tfrac{1}{2}\,\mathrm{gd}^{-1}\left(\frac{\pi}{2}x\right).$$

Solution (1) by K. B. Madhava and N. Sankara Aiyar;
(2) by T. P. Trivedi, M. A., LL. B.

(1) Now
$$\frac{1}{1^2-x^2}-\frac{3}{3^2-x^2}+\frac{5}{5^2-x^2}-\frac{7}{7^2-x^2}+\dots\dots$$

$$=\frac{\pi}{4}\left\{\frac{\pi}{\frac{\pi^2}{4}(1^2-x^2)}-\frac{3\pi}{\frac{\pi^2}{4}(3^2-x^2)}+\frac{5\pi}{\frac{\pi^2}{4}(5^2-x^2)}+\ldots\ldots\right\}.$$

$=\frac{\pi}{4}$. sec $\frac{\pi}{2}x$ (see Bromwich: *Infinite Series*, p 190, Ex. 19.)

Integrating each side we have

$$\tanh^{-1} x-\tanh^{-1}\frac{x}{3}+\tanh^{-1}\frac{x}{5}\ldots\ldots=\tfrac{1}{2}\,\mathrm{gd}^{-1}\left(\frac{\pi}{2}x\right).$$

Again, writing ix for x in the above

$$\frac{1}{1^2+x^2}-\frac{3}{3^2+x^2}+\frac{5}{5^2+x^2}-\frac{7}{7^2+x^2}+\ldots\ldots$$

$$=\frac{\pi}{4}\left\{\frac{\pi}{\frac{\pi^2}{4}(1^2+x^2)}-\frac{3\pi}{\frac{\pi^2}{4}(3^2+x^2)}+\frac{5\pi}{\frac{\pi^2}{4}(5^2+x^2)}\ldots\ldots\right\}$$

$$=\frac{\pi}{4}\mathrm{sech}\left(\frac{\pi}{2}x\right).$$

Integrating each side, we have

$$\tan^{-1} x-\tan^{-1}\frac{x}{3}+\tan^{-1}\frac{x}{5}-\tan^{-1}\frac{x}{7}+\ldots=\tfrac{1}{2}\,\mathrm{gd}\left(\frac{\pi}{2}x.\right)$$

(2) We have

$$\sin(x+iy)=(x+iy)\prod_1^\infty\left\{1-\frac{(x+iy)^2}{n^2\pi^2}\right\};$$

taking logarithms,

$$\log(\sin x\cosh y+i\sinh y\cos x)=\log(x+iy)+\log\left\{1-\frac{x+iy}{\pi}\right\}$$

$$+\log\left\{1+\frac{x+iy}{\pi}\right\}+\log\left\{1-\frac{x+iy}{2\pi}\right\}$$

$$+\log\left\{1+\frac{x+iy}{2\pi}\right\}+\ldots$$

Put $x=\frac{\pi}{4}$ and $y=\frac{\pi x}{4}$, and equate the imaginary parts on both sides, we get

$$\tan^{-1}\left(\tanh\frac{\pi}{4}x\right)=\tan^{-1}\frac{\frac{\pi x}{4}}{\frac{\pi}{4}}+\tan^{-1}\frac{-\frac{\pi x}{4}}{\frac{3\pi}{4}}+\tan^{-1}\frac{\frac{\pi x}{4}}{\frac{5\pi}{4}}+\text{etc.}$$

$$=\tan^{-1}x-\tan^{-1}\frac{x}{3}+\tan^{-}\frac{x}{5}-\tan^{-1}\frac{x}{7}+\text{etc.}$$

But $\quad 2\tan^{-1}\left(\tanh\frac{\pi}{4}x\right)=\tan^{-1}\dfrac{2\tanh\frac{\pi}{4}x}{1-\tanh^2\frac{\pi}{4}x}$

$$=\tan^{-1}\left(\sinh\frac{\pi}{2}x\right)=\mathrm{gd}\left(\frac{\pi}{2}x\right);$$

whence the first result follows.

Again

$$\frac{1}{2}\text{gd}^{-1}\left(\frac{\pi}{2}x\right)=\frac{1}{2}\log\ \tan\left(\frac{\pi}{4}+\frac{\pi}{4}x\right)$$

$$=\frac{1}{2}\left\{\log\ \sin\left(\frac{\pi}{4}+\frac{\pi}{4}x\right)-\log\cos\left(\frac{\pi}{4}+\frac{\pi}{4}\ \right)\right\}$$

$$=\frac{1}{2}\left\{\ \log\ \left(\frac{\pi}{4}+\frac{\pi}{4}x\right)+\log\left[1-\frac{\frac{\pi^2}{16}(1+x)^2}{\pi^2}\right]\right.$$

$$+\log\left[1-\frac{\frac{\pi^2}{16}(1+x)^2}{2^2\pi^2}\right]+\log\left[1-\frac{\frac{\pi^2}{16}(1+x)^3}{3^2\pi^2}\right]$$

$$+\text{etc}-\log\left[1-\frac{\frac{4\pi^2}{16}(1+x)^2}{\pi^2}\right]-\log\left[1-\frac{\frac{4\pi^2}{16}(1+x)^2}{3^2\pi^2}\right]$$

$$\left.-\log\left[1-\frac{\frac{4\pi^2}{16}(1+x)^2}{5^2\pi^2}\right]-\text{etc.}\right\}$$

$$=\frac{1}{2}\left\{\ \log\frac{\pi}{4}+\log\ (1+x)+\log(3-x)+\log(5+x)-\log 4^2\right.$$

$$+\log(7-x)+\log\ (9+x)-\log 8^2+\text{etc}-\log(3+x)$$

$$-\log\ (1+x)+\log 2^2-\log(5+x)-\log(7+x)$$

$$\left.+\log 6^2-\text{etc}\right\}$$

$$=\frac{1}{2}\left\{\ \log\frac{\pi}{4}+\log\frac{2^2.\ 6^2.\ 10^2\ldots\ldots}{4^2.\ 8^2.\ 12^2\ldots\ldots}+\log\frac{1+x}{1-x}\right.$$

$$\left.-\log\frac{3+x}{3-x}+\log\frac{5+x}{5-x}-\log\frac{7+x}{7-x}+\text{etc.}\right\}.$$

And

$$\log\frac{2^2.\ 6^2.\ 10^2\ldots\ldots}{4^2.\ 8^2.\ 12^2\ldots\ldots}=-\log\frac{2^2.\ 4^2.\ 6^2\ldots\ldots}{1^2.\ 3^2.\ 5^2\ldots\ldots}=-\log\frac{\pi}{4},$$

by employing Wallis' formula;

also

$$\frac{1}{2}\log\frac{1+x}{1-x}=\tanh^{-1}x;$$

$$\frac{1}{2}\log\frac{3+x}{3-x}=\frac{1}{2}\log\frac{1+\frac{x}{3}}{1-\frac{x}{3}}=\tanh^{-1}\frac{x}{3};\ \text{etc.}$$

$$\tanh^{-1}x-\tanh^{-1}\frac{x}{3}+\tanh^{-1}\frac{x}{5}-\text{etc}=\frac{1}{2}\text{gd}^{-1}\left(\frac{\pi}{2}x\right).$$

The second part may also be obtained readily by changing x into xi in the first.

Question 583.

(J. C. Swaminarayan, M. A.):—Show that

$$\int_0^{\frac{\sqrt{5}-1}{2}} \frac{\log(1+x)}{x}dx = \frac{\pi^2}{15} - \frac{1}{2}\left\{\log\frac{\sqrt{5}-1}{2}\right\}^2$$

and
$$\int_0^{\frac{\sqrt{5}-1}{2}} \frac{\log(1-x)}{x}dx = \left\{\log\frac{\sqrt{5}-1}{2}\right\}^2 - \frac{\pi^2}{10}.$$

(Suggested by Question 572 of Mr. Bhimasena Rao.)

Solution by K. B. Madhava and N. Sankara Aiyar, M. A.

Let
$$\phi(x) = x + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \frac{x^4}{4^2} + \ldots$$

Then
$$\phi(x) = \int^x -\frac{\log(1-x)}{x}\,dx.$$

and
$$\phi(-x) = \int^x -\frac{\log(1+x)}{x}\,dx.$$

so that we have

(i) $\phi(x) + \phi(-x) = \int^x -\frac{\log(1-x^2)}{x}dx = \frac{1}{2}\phi(x^2).$

Also we have

(ii) $\phi(x) + \phi(1-x) = \int\left\{\frac{\log x}{1-x} - \frac{\log(1-x)}{x}\right\}dx$

$$= \frac{\pi^2}{6} - \log x \log(1-x)$$

the constant being found by putting $x=1$.

Finally we have

(iii) $\phi(-x) + \phi\left(\frac{x}{1+x}\right) = \int -\frac{\log(1+x)}{x} + \frac{\log(1+x)}{x(1+x)}dx.$

$$= -\frac{1}{2}\left\{\log 1+x\right\}^2.$$

Now, if we put $x = \frac{1}{2}(\sqrt{5}-1)$, $x^2+x-1=0$;

$$1-x = \frac{x}{1+x}\,;\ (1+x) = \frac{1}{x},\text{ and } 1-x = x^2 \text{ etc.}$$

Hence

$$\phi(x) = \frac{\pi^2}{10} - (\log x)^2.$$

$$\phi(x^2) = \frac{\pi^2}{15} - (\log x)^2.$$

$$\phi(-x) = \frac{1}{2}(\log x)^2 - \frac{\pi^2}{15}.$$

which give the required results.

QUESTIONS FOR SOLUTION.

646. (R. SRINIVASAN, M.A.) :—If $xy+yz+zx=1$, show that

$$\sum\left\{(1+x^2)^{\frac{1}{2}}-x\right\}\left\{(1+y^2)^{\frac{1}{2}}-y\right\}=1.$$

647. (K. APPUKUTTAN ERADY, M.A.) :—The curves $\alpha=$constant, $\beta=$constant, where α, β are conjugate functions of (x, y) pass through a point. Prove that the curvatures of the two curves at the point are

$$-\frac{\partial h}{\partial \alpha} \text{ and} -\frac{\partial h}{\partial \beta},$$

where

$$h=\left\{\frac{\partial(\alpha,\beta)}{\partial(x,y)}\right\}^{\frac{1}{2}}$$

648. (K. APPUKUTTAN ERADY, M.A.) :—Tangents TP, TQ are drawn to the ellipse $x^2/a^2+y^2/b^2=1$, and the parallelogram TPSQ is completed. Prove that the area of the locus of S, as T completes a circuit along $x^2/\lambda^2+y^2/\mu^2=1$, is

$$\pi\lambda\mu+2\pi\, ab(a^2/\lambda^2+b^2/\mu^2-2).$$

649. (S. MALHARI RAO, B.A.) :—A gentleman has one third of an acre of vacant land round his house. He wishes to divide it into five different plots to grow five different kinds of flowers. What must be the areas of these plots so that they may be in A. P. and each of them may be contained an exact number of times in an acre? Shew that there cannot be more than one set of values for the areas.

650. (N. P. PANDYA) :—Circumscribe an ellipse about a given triangle so that the centre of the ellipse may coincide with the in-centre of the triangle.

651. (S. MALHARI RAO, B. A.) :—The Nasik or pan-diagonal Magic Square (a) has been constructed by placing the series of natural numbers from 1 upto 25 along the combined paths of a Knight and a Rook in chess. By adopting another method, and guided by the altered positions of 1, 7, 13, 19 and 25 as shown in square (b), fill up the vacant cells in it so that it may be magic and pan-diagonal.

(a)

12	25	8	16	4
18	1	14	22	10
24	7	20	3	11
5	13	21	9	17
6	19	2	15	23

(b)

13				
	19			
		25		
			1	
				7

652. (S. P. SINGARAVELU MUDALIAR):—From a point (eccentric angle ϕ) of an ellipse of semi—axes a, b, three normals are drawn to the ellipse, shew that the square on the radius of the circle passing through the feet of the normals is

$$\left(a+\frac{b^2}{2a}\right)^2\cos^2\phi+\left(b+\frac{a^2}{2b}\right)^2\sin^2\phi.$$

653. (Selected):—Find a cube the sum of whose divisors is a perfect square.

654. (R. VYTHYNATHASWAMY):—A plane moves in any manner on a fixed plane and returns to its original position after n revolutions. If A, B, C, D be the areas described by four points a, b, c, d in the moving plane, prove the identical relation

$$\begin{vmatrix} 0 & 1 & 1 & 1 & 1 & n\pi \\ 1 & 0 & ab^2 & ac^2 & ad^2 & A \\ 1 & ba^2 & 0 & bc^2 & bd^2 & B \\ 1 & ca^2 & cb^2 & 0 & cd^2 & C \\ 1 & da^2 & db^2 & dc^2 & 0 & D \\ n\pi & A & B & C & D & 0 \end{vmatrix}=0.$$

Shew that if we put $\frac{1}{2}\pi$ for $n\pi$ in this, we get the identical relation between the mutual distances of the four points and the areas of their pedals with respect to any closed curve.

Also prove from this relation that if $A=B=C=D$, then the four points lie on a circle.

655. (A. NARASINGA RAO):—Solve for f from the relation

$$f^{(n)}(x)=\alpha f(x+\beta)$$

α and β being any constants.

A curve and its nth evolute are similar. Find the intrinsic equation of the curve.

656. (A. NARASINGA RAO):—A square is divided into n^2 smaller squares by lines || to the sides. P,Q are any two points within the square. What is the average number of lines which the finite line PQ meets? What is the chance that it should meet just k lines?

657. (T. P. TRIVEDI, M.A., LL.B.):—Prove that the value of the determinant of the $(n+2)^{th}$ order

$$\begin{vmatrix} 0, & 1, & 1, & 1, & \dots & \dots & 1 \\ 1, & 0, & a_1+a_2, & a_1+a_3 & \dots & \dots & a_1+a_{n+1} \\ 1, & a_1+a_2 & 0, & a_2+a_3 & \dots & \dots & a_2+a_{n+1} \\ 1, & a_1+a_{n+1}, & a_2+a_{n+1}, & a_3+a_{n+1} & \dots & \dots & 0 \end{vmatrix}$$

is equal to

$$-(-2)^n a_1 a_2 a_3 \ldots a_{n+1} \left\{ \frac{1}{a_1} + \frac{1}{a_2} + \ldots \frac{1}{a_{n+1}} \right\}.$$

Also find the value of the determinant of the $(n+1)^{th}$ order, obtained from the above by deleting the first column and the first row.

658. (K. J. Sanjana, M. A.) :—Prove the identity

$$2\{ax+by+(a+b)(x+y)\}^3$$
$$-6\{ax+by+(a+b)(x+y)\}(a^2+ab+b^2)(x^2+xy+y^2)$$
$$-27abxy(a+b)(x+y)=(a-b)(x-y)(2a+b)(a+2b)(2x+y)(x+2y);$$

and deduce the theorem of Cayley that

$$ax+by+cz,\ bx+cy+az,\ cx+ay+bz$$

are the three roots of the equation

$$4t^3-3(a^2+b^2+c^2)(x^2+y^2+z^2)t-54\,abcxyz$$
$$-2(b-c)(c-a)(a-b)(y-z)(z-x)(x-y)=0,$$

when $a+b+c$ and $x+y+z$ both vanish.

[In the *Messenger of Mathematics* for November 1914, Prof. Steggall has given a direct proof of the theorem which, it appears, was proved by Cayley by the use of determinants. The question suggests a very simple indirect proof.]

659. (M. Bhimasena Rao) :—Shew that

$$(1)\ \left(1+e^{-\pi}\right)\left(1+e^{-2\pi}\right)\left(1+e^{-3\pi}\right)\ldots\ldots=e^{\pi/24}\left(\frac{1}{2}\right)^{\frac{1}{8}}$$

$$(2)\ \left(1+e^{-\pi\sqrt{2}}\right)\left(1+e^{-2\pi\sqrt{2}}\right)\left(1+e^{-3\pi\sqrt{2}}\right)\ldots=$$
$$e^{\pi\sqrt{2}/24}\left(\frac{1}{2}\right)^{\frac{1}{4}}.$$

660. (M. Bhimasena Rao) :—Shew that the locus of the double point of orthologic triangles described on AB and CD is the radical axis of the circles having AD and BC as diameters.

661. (S. Ramanujan) :—Solve in integers

$$x^3+y^3+z^3=u^6,$$

and deduce the following:

$$6^3-5^3-3^3=2^6;$$
$$8^3+6^3+1^3=3^6;$$
$$12^3-10^3-1^3=3^6;$$
$$46^3-37^3-3^3=6^6;$$
$$174^3+133^3-45^3=14^6;$$
$$1188^3-509^3-3^3=34^6.$$

662. (S. Ramanujan) :—Let AB be a diameter and BC be a chord of a circle ABC. Bisect the minor arc BC at M; and draw a chord BN equal to half of the chord BC. Join AM. Describe two

circles with A and B as centres and AM and BN as radii cutting each other at S and S′ and cutting the given circle again at the points M′ and N′ respectively. Join AN and BM intersecting at R and also join AN′ and BM′ intersecting at R′. Through B draw a tangent to the given circle meeting AM and AM′ produced at Q and Q′ respectively. Produce AN and M′B to meet at P and also produce AN′ and MB to meet at P′. Show that the eight points PQRSS′R′Q′P′ are cyclic and that the circle passing through these points is orthogonal to the given circle ABC.

663. (S. Krishnaswami Aiyangar, B. A.):—Establish the following relation with respect to Bernoulli's numbers :—

$$\frac{n}{2}=c_2.\, B_1-c_4.\, B_3.\, 4+c_6.\, B_5.\, 4^2-(-)^{n-1}c_{2n}.\, B_{2n-1}.4^{n-1}$$

where $c_r={}_{2n+1}C_r$

664. (S. Krishnaswami Aiyangar, B. A.) :—Prove that

$$\sum_1^{\infty}\left\{\frac{\Gamma(n+1)}{\Gamma(n+\frac{3}{2})}\right\}^2.\frac{1}{n}=\frac{16}{\pi}-4.$$

665. (R. N. Apte, M.A., F.R.A.S.) :—Find the value of

$$\iint xyz\left\{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right\}^{\frac{1}{2}}dx\,dy$$

where $z\equiv(1-x^2/a^2-y^2/b^2)^{\frac{1}{2}}$ and the integration is over the positive quadrant of $x^2/a^2+y^2/b^2=1$.

666. (S. Ramanujan) :—Solve in positive rational numbers

$$x^y=y^x.$$

For example : $x=4,\ y=2$; $x=3\frac{3}{8},\ y=2\frac{1}{4}$.

667. (S. Narayana Aiyar, M. A.) :—Establish the truth of the following theorems :

1. If $F(y)=\phi(y)=\frac{x}{1!}\phi(y+1)+\frac{x^2}{2!}\phi(y+2)-\frac{x^3}{3!}\phi(y+3)+\ldots$

then $\phi(y)=F(y)+\frac{x}{1!}F(y+1)+\frac{x^2}{2!}F(y+2)+\frac{x^3}{3!}F(y+3)+\ldots$

2. If $F(y)=\phi(y)-\frac{n}{1}\phi(y+1)+\frac{n(n+1)}{1\cdot 2}\phi(y+2)-\ldots$

then $\phi(y)=F(y)+\frac{n}{1}.F(y+1)+\frac{n(n-1)}{1\cdot 2}F(y+2)+\ldots$